Math 54 Handout 21

August 3, 2018

Question 1.

Solve the following heat flow problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}; \quad 0 \leq x \leq L \\ u(0,t) = 0, \quad u(L,t) = 1 \\ u(x,0) = 0 \end{cases}$$

We seek solutions of the form $w(x) + v(x,t)$ where $w(x)$ is a time-independent solution and $v(x,t)$ is "transient". $w(x)$ satisfies the following

$$\begin{cases} w''(x) = 0 \\ w(0) = 0, w(L) = 1 \end{cases}$$

Solving for $w(x)$ we get $w(x) = \frac{x}{L}$. Now $v(x,t)$ satisfies

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}; \quad 0 \leq x \leq L \\ v(0,t) = 0, \quad v(L,t) = 0 \\ v(x,0) = -w(x) = -\frac{x}{L} \end{cases}$$

The solution is of the form

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi}{L}\right)^2 \beta t} sin\left(\frac{n\pi x}{L}\right)$$

so all we need to do is find the Fourier sine series of $-\frac{x}{L}$.

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L -\frac{x}{L} sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L^2}\frac{L}{n\pi} x cos\left(\frac{n\pi x}{L}\right)\bigg|_0^L - \frac{2}{L^2}\frac{L}{n\pi}\int_0^L cos\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{2}{n\pi} cos(n\pi) = (-1)^n \frac{2}{n\pi} \end{aligned}$$

so we see that

$$v(x,t) = \sum_{n=1}^{\infty} (-1)^n \frac{2}{n\pi} e^{-\left(\frac{n\pi}{L}\right)^2 \beta t} sin\left(\frac{n\pi x}{L}\right)$$

and hence the solution is

$$u(x,t) = w(x) + v(x,t) = \frac{x}{L} + \sum_{n=1}^{\infty} (-1)^n \frac{2}{n\pi} e^{-\left(\frac{n\pi}{L}\right)^2 \beta t} sin\left(\frac{n\pi x}{L}\right)$$

Question 2.

Solve the following heat flow problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}; \quad 0 \le x \le L_1, 0 \le y \le L_2 \\ u(x,0,t) = u(x,L_2,t) = 0 \\ u(0,y,t) = u(L_1,y,t) = 0 \\ u(x,y,0) = 1 \end{cases}$$