Math 54 Handout 20

August 3, 2018

Question 1.

Compute the Fourier series for

$$f(x) = |x|, \quad -L \le x \le L$$

Since $|x|$ is an even function, automatically all coefficient for the sine part is 0, so we only need to compite the cosine coefficient.

$$a_0 = \frac{1}{L}\int_{-L}^{L} |x|dx = \frac{2}{L}\int_{0}^{L} xdx = \frac{2}{L}\frac{1}{2}x^2\Big|_0^L = L$$

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} |x|cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L xcos\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{2}{L}\frac{xL}{n\pi}sin\left(\frac{n\pi x}{L}\right)\Big|_0^L - \frac{2}{L}\int_0^L \frac{L}{n\pi}sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\left(\frac{L}{n\pi}\right)^2 cos\left(\frac{n\pi x}{L}\right)\Big|_0^L \\ &= \frac{2L}{(n\pi)^2}(cos(n\pi) - 1) = \begin{cases} \frac{-4L}{(n\pi)^2} & \text{n odd} \\ 0 & \text{n even} \end{cases} \end{aligned}$$

So we see that

$$f(x) = |x| = \frac{L}{2} + \sum_{odd} \frac{-4L}{(n\pi)^2} cos\left(\frac{n\pi x}{L}\right)$$

Question 2.

Compute the Fourier cosine series and Fourier sine series for

$$f(x) = L - x, \quad 0 \le x \le L$$

The Fourier cosine series for $-x$ is

$$-\frac{L}{2} + \sum_{odd} \frac{4L}{(n\pi)^2} cos\left(\frac{n\pi x}{L}\right)$$

, and so the Fourier cosine series for $L - x$ is

$$L - \frac{L}{2} + \sum_{odd} \frac{4L}{(n\pi)^2} cos\left(\frac{n\pi x}{L}\right) = \frac{L}{2} + \sum_{odd} \frac{4L}{(n\pi)^2} cos\left(\frac{n\pi x}{L}\right)$$

Question 3.

Solve the following heat flow problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = 1 \end{cases}$$

The solution is of the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi}{L}\right)^2 \beta t} sin\left(\frac{n\pi x}{L}\right)$$

so all we need to do is find the Fourier sine series of 1. We compute

$$b_n = \frac{2}{L}\int_0^L sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L}\frac{-L}{n\pi} cos\left(\frac{n\pi x}{L}\right)\Big|_0^L = \frac{2}{n\pi}(1 - cos(n\pi)) = \begin{cases} \frac{4}{n\pi} & \text{n odd} \\ 0 & \text{n even} \end{cases}$$

so we see that the solution is

$$u(x,t) = \sum_{odd} \frac{4}{n\pi} e^{-\left(\frac{n\pi}{L}\right)^2 \beta t} sin\left(\frac{n\pi x}{L}\right)$$