Math 54 Handout 19

August 3, 2018

Question 1.

1. The polynomial is $x^2 + \lambda = 0$, and so the roots are $\pm\sqrt{-\lambda}$.

   For $\lambda < 0$, the solution is of the form $y(t) = c_1 e^{\sqrt{-\lambda}t} + c_2 e^{-\sqrt{-\lambda}t}$. Plugging in the boundary condition, we have the following

$$\begin{cases} c_1 + c_2 = c_1 e^{\sqrt{-\lambda}2\pi} + c_2 e^{-\sqrt{-\lambda}2\pi} \\ \sqrt{-\lambda}c_1 - \sqrt{-\lambda}c_2 = \sqrt{-\lambda}c_1 e^{\sqrt{-\lambda}2\pi} - \sqrt{-\lambda}c_2 e^{\sqrt{-\lambda}2\pi} \end{cases}$$

   putting it into a matrix we have

$$\begin{pmatrix} 1 - e^{\sqrt{-\lambda}2\pi} & 1 - e^{-\sqrt{-\lambda}2\pi} \\ \sqrt{-\lambda}(1 - e^{\sqrt{-\lambda}2\pi}) & -\sqrt{-\lambda}(1 - e^{\sqrt{-\lambda}2\pi}) \end{pmatrix} \to \begin{pmatrix} 1 - e^{\sqrt{-\lambda}2\pi} & 1 - e^{-\sqrt{-\lambda}2\pi} \\ 0 & -2\sqrt{-\lambda}(1 - e^{\sqrt{-\lambda}2\pi}) \end{pmatrix}$$

   which has 2 pivots because $1 - e^{\sqrt{-\lambda}2\pi} \neq 0$. Thus $c_1 = c_2 = 0$ and there is only the trivial solution.

   For $\lambda = 0$, the solution is of the form $y(t) = c_1 + c_2 t$. Plugging in the boundary condition, we have the following

$$\begin{cases} c_1 = c_1 + c_2 2\pi \\ c_2 = c_2 \end{cases}$$

   and the solution is $c_2 = 0$, and c_1 can be anything, so we see that $y(t) = c_1$ a constant is a solution.

   For $\lambda > 0$, the solution is of the form $y(t) = c_1 sin(\sqrt{\lambda}t) + c_2 cos(\sqrt{\lambda}t)$. Plugging in the boundary condition gives

$$\begin{cases} c_2 = c_1 sin(\sqrt{\lambda}2\pi) + c_2 cos(\sqrt{\lambda}2\pi) \\ \sqrt{\lambda}c_1 = \sqrt{\lambda}c_1 cos(\sqrt{\lambda}2\pi) - c_2\sqrt{\lambda}sin(\sqrt{\lambda}2\pi) \end{cases}$$

   putting it into a matrix we have

$$\begin{pmatrix} sin(\sqrt{\lambda}2\pi) & cos(\sqrt{\lambda}2\pi) - 1 \\ \sqrt{\lambda}(cos(\sqrt{\lambda}2\pi) - 1) & -\sqrt{\lambda}sin(\sqrt{\lambda}2\pi) \end{pmatrix}$$

   The determinant of the above matrix is

$$-\sqrt{\lambda}sin^2(\sqrt{\lambda}2\pi) - \sqrt{\lambda}(cos^2(\sqrt{\lambda}2\pi) - 2cos(\sqrt{\lambda}2\pi) + 1)$$

   simplifying (note $sin^2 + cos^2 = 1$) gives

$$-2\sqrt{\lambda} + 2\sqrt{\lambda}cos(\sqrt{\lambda}2\pi)$$

   We want the determinant to be 0 (because we don't want 2 pivots for the matrix, since it will imply trivial solution), so we want

$$1 = cos(\sqrt{\lambda}2\pi)$$

In order for cosine to be 1, the insides must be $2n\pi$, and so we see that $\sqrt{\lambda}2\pi = 2n\pi$, and hence $\sqrt{\lambda} = n$, i.e. $\lambda = n^2$. The eigenvalues are n^2, and plugging $\sqrt{\lambda} = n$ to the matrix we obtain the zero matrix, so we see that for $\sqrt{\lambda} = n$, both c_1 and c_2 are free, so the eigenfunctions are

$$c_1 sin(nt) + c_2 cos(nt)$$

2. This example was done in lecture.

Question 2.

Compute the Fourier series for

$$f(x) = \begin{cases} -1, & -\pi \leq x \leq 0 \\ 1, & 0 \leq x \leq \pi \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = 0 \quad \text{because f(x) is odd}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(nx)dx = 0 \quad \text{because inside is odd}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(nx)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)sin(nx)dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} sin(nx)dx = \frac{2}{n\pi}(-cos(nx))\Big|_0^{\pi} = \frac{2}{n\pi}(1 - cos(n\pi)) \\ &= \begin{cases} \frac{4}{n\pi} & \text{n odd} \\ 0 & \text{n even} \end{cases} \end{aligned}$$

So we see that

$$f(x) = \sum_{odd} \frac{4}{n\pi} sin(nx)$$