Math 54 Handout 15

July 27, 2018

Question 1.

Find the singular value decomposition $A = U\Sigma V^T$ of $A = \begin{pmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{pmatrix}$

$A^TA = \begin{pmatrix} 81 & -27 \\ -27 & 9 \end{pmatrix}$ which has characteristic polynomial $(81-x)(9-x) - 27^2 = x^2 - 90x$. The eigenvalues are 0 and 90. For the eigenvalue $\lambda_1 = 90$, we have eigenvector $v_1 = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$, and the eigenvalue $\lambda_2 = 0$ corresponds to eigenvector $v_2 = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$. We found that $V = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}$.

Now $Av_1 = \frac{1}{\sqrt{10}}\begin{pmatrix} -10 \\ 20 \\ 20 \end{pmatrix}$ which has length $|Av_1| = \sqrt{90}$. Since $\lambda_2 = 0$, we don't consider Av_2. We see that $u_1 = \frac{1}{3}\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$. Now we choose any two unit vectors u_2, u_3 such that $\{u_1, u_2, u_3\}$ is orthonormal. We choose $u_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $u_3 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$. We conclude that $U = \begin{pmatrix} \frac{-1}{3} & \frac{2}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & 0 \\ \frac{2}{3} & 0 & \frac{1}{\sqrt{5}} \end{pmatrix}$. Finally, $\Sigma = \begin{pmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Thus we see that the singular value decomposition of A is

$$\begin{pmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{pmatrix} = \begin{pmatrix} \frac{-1}{3} & \frac{2}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & 0 \\ \frac{2}{3} & 0 & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix}^T$$

Question 2.

Let $A = U\Sigma V^T$ be the singular value decomposition of A. Show that the columns of V are the eigenvectors of A^TA, and the columns of U are the eigenvectors of AA^T.

$A^TA = V\Sigma^T\Sigma V^T$ is an orthogonal diagonalization of A^TA since $\Sigma^T\Sigma$ is a diagonal matrix and V is orthogonal. Thus the columns of V are the orthonormal eigenvectors of A^TA. Similarly, $AA^T = U\Sigma\Sigma^TU^T$.

As $\Sigma\Sigma^T$ is diagonal and U is orthogonal, we see that the columns of U are the orthonormal eigenvectors of AA^T.

Question 3.

Show that if P is an orthogonal $m \times m$ matrix, then PA has the same singular values as A.

The singular values of A are the square roots of the eigenvalues of $A^T A$. Since $(PA)^T(PA) = A^T P^T PA = A^T A$, we conclude that A and PA have the same singular values.