# Math 54 Handout 14

July 20, 2018

### Question 1.

$\text{Proj}_q p = \frac{\langle p,q\rangle}{|q|^2} q = \frac{28}{27} q = \frac{28}{27}(5 - 4t^2)$.

### Question 2.

Consider the inner product space $C^\infty[-1,1]$ with inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt$$

Show that $S = \{\frac{1}{\sqrt{2}}, sin(2\pi t), cos(2\pi t)\}$ is an orthonormal set, and compute the projection of $t$ on Span$\{S\}$.

$$\int_{-1}^{1} \frac{1}{\sqrt{2}} sin(2\pi t)dt = 0 \text{ (because sine is an odd function)}$$

$$\int_{-1}^{1} \frac{1}{\sqrt{2}} cos(2\pi t)dt = \frac{1}{\sqrt{2}}\frac{1}{2\pi} sin(2\pi t)\Big|_{-1}^{1} = 0$$

$$\int_{-1}^{1} sin(2\pi t) * cos(2\pi t)dt = 0 \text{ (because the inside is an odd function)}$$

$$\int_{-1}^{1} \left(\frac{1}{\sqrt{2}}\right)^2 dt = 1$$

$$\int_{-1}^{1} cos^2(2\pi t)dt = \int_{-1}^{1} \frac{1}{2}(cos(2\pi t) + 1)dt = \frac{1}{2}\left(\frac{1}{2\pi} sin(2\pi t) + t\right)\Big|_{-1}^{1} = 1$$

$$\int_{-1}^{1} sin^2(2\pi t)dt = \int_{-1}^{1} \frac{1}{2}(1 - cos(2\pi t))dt = \frac{1}{2}\left(t - \frac{1}{2\pi} sin(2\pi t)\right)\Big|_{-1}^{1} = 1$$

so the set $S = \{\frac{1}{\sqrt{2}}, sin(2\pi t), cos(2\pi t)\}$ is an orthonormal set.

Now we compute the projection of $t$ on Span$\{S\}$.

$$\left\langle t, \frac{1}{\sqrt{2}}\right\rangle = \int_{-1}^{1} \frac{t}{\sqrt{2}} dt = 0 \text{ (because t is an odd function)}$$

$$\langle t, sin(2\pi t)\rangle = \int_{-1}^{1} t sin(2\pi t)dt = -t\frac{1}{2\pi} cos(2\pi t)\Big|_{-1}^{1} + \frac{1}{2\pi}\int_{-1}^{1} cos(2\pi t)dt = \frac{-1}{\pi}$$

$$\langle t, cos(2\pi t)\rangle = \int_{-1}^{1} tcos(2\pi t)dt = 0 \text{ (because the inside is an odd function)}$$

so the projection of $t$ on Span$\{S\}$ is just $\frac{-1}{\pi}sin(2\pi t)$.

## Question 3.

The eigenvalues are solutions to the polynomial $(2-\lambda)(4-\lambda)-(-\sqrt{3})(-\sqrt{3}) = 8-6\lambda+\lambda^2-3 = \lambda^2-6\lambda+5 = (\lambda-1)(\lambda-5)$, so the eigenvalues are 1 and 5. For $\lambda = 1$, we find a basis of the null space of $\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix}$, which is $v_1 = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$, while for $\lambda = 5$, we find a basis of the null space of $\begin{pmatrix} -3 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$, which is $v_2 = \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}$. Normalizing $v_1$ and $v_2$ gives $v_1 = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$ and $v_2 = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$. Thus

$$\begin{pmatrix} 2 & -\sqrt{3} \\ -\sqrt{3} & 4 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

## Question 4.

True or False:

1. False. $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

2. True. Since $A$ is invertible, the diagonals of $D$ are nonzero (denote the diagonal by $\lambda_1, ..., \lambda_n$), and hence $D$ is invertible with inverse $D^{-1}$ consisting of diagonal $\frac{1}{\lambda_1}, ..., \frac{1}{\lambda_n}$. If $A = UDU^T$, then $A^{-1} = UD^{-1}U^T$ which is orthogonally diagonalizable.

## Question 5.

If $A$ and $B$ are both symmetric, then $(AB)^T = B^T A^T = BA = AB$, so $AB$ is also symmetric.

## Question 6.

$(Ax) \cdot y = x^T A^T y = x^T Ay = x \cdot (Ay)$.