Math 54 Handout 13

July 17, 2018

Question 1.

1. Since the columns of U are orthonormal, automatically $U^TU = I$. On the other hand, $UU^T = \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{pmatrix}$

2. $\text{Proj}_W y = UU^T y = \frac{1}{9} \begin{pmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ 8 \\ 1 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 18 \\ 36 \\ 45 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$

Question 2.

Let W be a subspace of $\mathbb{R}^n$ with an orthogonal basis $\{w_1, ..., w_p\}$ and let $\{v_1, ..., v_q\}$ be an orthogonal basis for $W^\perp$.

1. The w_i's are orthogonal to each other, and the v_j's are also orthogonal to each other. We only need to check that w_i and v_j are orthogonal. However, since $w_i \in W$ and $v_j \in W^\perp$, we know that they are. Thus $\{w_1, ..., w_p, v_1, ..., v_q\}$ is an orthogonal set.

2. Every vector $v \in \mathbb{R}^n$ can be written as $\hat{v} + y$ where $\hat{v} \in W$ and $y \in W^\perp$, so $\{w_1, ..., w_p, v_1, ..., v_q\}$ spans $\mathbb{R}^n$.

3. Since $\{w_1, ..., w_p, v_1, ..., v_q\}$ is linearly independent and span $\mathbb{R}^n$, we know that it is a basis, so we see that $dim(W) + dim(W^\perp) = p + q = n$.

Question 3.(Originally the problem is not complete, so I changed it slightly)

Find an orthogonal basis for the column space of the matrix

$$\begin{pmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{pmatrix}$$

We apply the Gram-Schmidt method. Let $w_1 = \begin{pmatrix} 3 \\ 1 \\ -1 \\ 3 \end{pmatrix}$

$$w_2 = \begin{pmatrix} -5 \\ 1 \\ 5 \\ -7 \end{pmatrix} - (-2) \begin{pmatrix} 3 \\ 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 3 \\ -1 \end{pmatrix}$$

$$w_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \\ 8 \end{pmatrix} - \frac{3}{2} \begin{pmatrix} 3 \\ 1 \\ -1 \\ 3 \end{pmatrix} - \frac{-1}{2} \begin{pmatrix} 1 \\ 3 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 1 \\ 3 \end{pmatrix}$$

Question 4.

$$A^T A = \begin{pmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 6 & -11 \\ -11 & 22 \end{pmatrix}$$

$$A^T b = \begin{pmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 11 \end{pmatrix}$$

so the solution to $A^T A x = A^T b$ is $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$