Math 54 Handout 12

July 13, 2018

Question 1.

If $v \in W \cap W^{\perp}$, then v is orthogonal to itself, and hence $v \cdot v = 0$. But this implies $v = 0$.

Question 2.

Checking they are orthogonal is checking the dot product between distinct vectors. For example

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} = -1 + 0 + 1 = 0$$

We want to now express $\begin{pmatrix} 8 \\ -4 \\ -3 \end{pmatrix}$ as a linear combinations of elements in the orthogonal set. To do this we calculate

$$\frac{v \cdot u_1}{|u_1|^2} = \frac{5}{2}, \frac{v \cdot u_2}{|u_2|^2} = \frac{-27}{18} = \frac{-3}{2}, \frac{v \cdot u_3}{|u_3|^2} = \frac{18}{9} = 2$$

so $v = \frac{5}{2}u_1 + \frac{-3}{2}u_2 + 2u_3$.

Question 3.

We can either check that the columns are orthonormal, or we can just brute force multiply $U^T U$. For both cases, we we multiply $U^T T$, we get

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & sin(\theta) \\ 0 & -sin(\theta) & cos(\theta) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Question 4.

Since $U^T U = I$, we see that $det(U^T U) = det(U^T)det(U) = det(U)^2 = 1$, so $det(U) = \pm 1$.

Question 5.

$(Ax) \cdot (Ax) = (Ax)^T (Ax) = x^T A^T A x = x^T x = x \cdot x$.

Question 6.

Let W be the subspace spanned by the $u's$ and write y as a sum of a vector in W and a vector orthogonal to W, where

$$y = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}, u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, u_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$$

$\text{Proj}_W y = \text{Proj}_{u_1} y + \text{Proj}_{u_2} y = 2u_1 + \frac{1}{2}u_2 = \begin{pmatrix} \frac{3}{2} \\ \frac{7}{2} \\ 1 \end{pmatrix}$.

Now $y - \text{Proj}_W y = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} - \begin{pmatrix} \frac{3}{2} \\ \frac{7}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{5}{2} \\ \frac{1}{2} \\ 2 \end{pmatrix}$