MATH 256 A: PROBLEM SET 1, DUE WED SEP 7

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(1) Let $\mathcal{C}$ be a category. Recall that for an object $A \in \mathcal{C}$ we defined a functor

$$h_A : \mathcal{C}^{op} \to \mathrm{Set}, \quad B \mapsto \mathrm{Hom}_{\mathcal{C}}(B, A).$$

Prove Yoneda's lemma: The association $A \mapsto h_A$ defines a fully faithful functor

$$h : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set}).$$

(2) Let k be a ring, and let $f_1(\underline{X}), \dots, f_r(\underline{X}) \in k[\underline{X}] = k[X_1, \dots, X_m]$ be polynomials. For a morphism of rings $h : k \to R$ we then get polynomials $f_i^h \in R[\underline{X}]$ by taking the image of f_i under the map $k[\underline{X}] \to R[\underline{X}]$ induced by h. Let

$$F : (k\text{-algebras}) \to \mathrm{Set}$$

be the functor sending a k-algebra $h : k \to R$ to the set

$$\{(a_1, \dots, a_m) \in R^m | f_i^h(\underline{a}) = 0 \quad \text{for all } i\}.$$

Show that there exists a k-algebra S and an isomorphism of functors $\iota : h_S^{op} \simeq F$, where h_S^{op} is the functor sending a k-algebra R to $\mathrm{Hom}(S, R)$. Moreover, the pair (S, ι) is unique up to unique isomorphism.

(3) Let $\mathcal{C}$ be a category, and let $f : X \to Z$ and $g : Y \to Z$ be two morphisms. Let

$$F : \mathcal{C}^{op} \to \mathrm{Set}$$

be the functor

$$W \mapsto \{(a : W \to X, b : W \to Y) | f \circ a = g \circ b\}.$$

(i) Suppose that F is representable, and let (S, ι) be a pair with $S \in \mathcal{C}$ and $\iota : h_S \simeq F$. Show that there is a commutative diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z. \end{tikzcd}$$

The category is said to have *fiber products* if the functor F is representable for all morphisms f and g.

(ii) Show that the category of sets has fiber products. What about Set^{op}?

(iii) Does the category Ring of commutative rings have fiber products?

(iv) How about the category Ring^{op}?

(4) Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms, and let $I \subset B$ be an ideal of A. Let $I^e \subset B \otimes_A C$ be the extension of I to $B \otimes_A C$ (the image of $I \otimes_A C \to B \otimes_A C$). Show that there is a natural isomorphism

$$(B/I) \otimes_A C \simeq (B \otimes_A C)/I.$$

Hint: Use Yoneda's lemma and think about the functors each side represents.

(5) (Inverse limits) Let Gp denote the category of groups, and suppose given a sequence

$$\cdots \longrightarrow G_n \xrightarrow{\pi_n} G_{n-1} \longrightarrow \cdots \xrightarrow{\pi_4} G_3 \xrightarrow{\pi_3} G_2 \xrightarrow{\pi_2} G_1$$

of morphisms in Gp. Let

$$\varprojlim G_n : \mathrm{Gp}^{op} \to \mathrm{Set}$$

be the functor sending a group H to the set collections of maps $\{h_n : H \to G_n\}_{n=1}^{\infty}$ such that for all $n \geq 2$ we have $h_{n-1} = \pi_n \circ h_n$. Show that $\varprojlim G_n$ is representable (the representing object is called the *inverse limit*).