Math 256A Problem Set 9

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Question 1.

(a) Suppose $\mathscr{F}'$ and $\mathscr{F}''$ are locally free, and suppose $U = \text{Spec } A$ is an affine open set where $\mathscr{F}'|_{\text{Spec } A} = \tilde{A}^{\oplus p}$ and $\mathscr{F}''|_{\text{Spec } A} = \tilde{A}^{\oplus q}$, then we see that given any $V \subset U$, we have an exact sequence

$$0 \to \tilde{A}^{\oplus p}(V) \to \mathscr{F}|_{\text{Spec } A}(V) \to \tilde{A}^{\oplus q}(V) \to 0$$

However, since all short exact sequences where the last term is free splits, we see that the above short exact sequence splits, and hence $\mathscr{F}|_{\text{Spec } A}(V) \cong \tilde{A}^{\oplus(p+q)}(V)$ is free for any $V \subset U$, and $\mathscr{F}|_{\text{Spec } A} = \tilde{A}^{\oplus(p+q)}$. Thus we see that $\mathscr{F}$ is also locally free.

(b) Suppose $\mathscr{F}$ and $\mathscr{F}''$ are locally free of finite rank. As all sheaves are assumed quasicoherent, we reduce the case to $X = \text{Spec } A$, where $\mathscr{F} = \tilde{A}^{\oplus m}$ and $\mathscr{F}'' = \tilde{A}^{\oplus n}$. Consider the map $\phi : \mathscr{F} \to \mathscr{F}''$ appearing in the exact sequence. $\phi(X) : A^{\oplus m} = \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}'') = A^{\oplus n}$ is as a $n \times m$ matrx with entries in A, and hence $\phi : \mathscr{F} \to \mathscr{F}''$ can be interpreted as a $n \times m$ matrx M with entries in A. At each point p, since exactness of the original sequence is the same as exactness on the level of stalks, we know that the stalk at p is exact, and so the map $\mathscr{F}_p \to \mathscr{F}''_p$ is surjective. This map is given by $(M)_p$ by localizing each entries of M at p. Thus surjectiveness show that $(M)_p$ has n linearly independent columns, and hence M has n linearly independent columns. Now as the $n \times n$ minor is non-vanishing at p, it is non-vanishing near p because determinant is a continuous function. Thus we can cover X with distinguished open sers in bijection with the choices of n columns of M. Restricting to one such subset, we can rename the columns so that M has the first n columns linearly independent. Now multiplying the original coordinates by the inverse of the $n \times n$ matrix in M times the determinant of that $n \times n$ matrix and using the resulting coordinates to replace the original coordinates. After this change of coordinates, M now has the identity matrix in the first n columns and 0 otherwise. Thus this allows us to interpret $\mathscr{F}'$ as the kernel, $\widetilde{A^{\oplus(m-n)}}$.

(c) Even if $\mathscr{F}'$ and $\mathscr{F}$ are both locally free, $\mathscr{F}''$ need not be. To show this, consider

$$0 \to tk[t] \to k[t] \to k[t]/(t) \to 0$$

a short exact sequence of $k[t]$-module. $tk[t]$ and $k[t]$ are free module over $k[t]$ generated by t and 1 respectively, but $k[t]/(t) \cong k$ is not a free $k[t]$-module. Thus even if $\mathscr{F}'$ and $\mathscr{F}$ are both locally free, $\mathscr{F}''$ need not be.

Question 2.

Suppose $\mathscr{F}$ and $\mathscr{G}$ are quasicoherent sheaves, and suppose on an affine open U, $\Gamma(U, \mathscr{F}) = M$ and $\Gamma(U, \mathscr{G}) = N$, then by the construction of tensor product, $\Gamma(U, \mathscr{F} \otimes \mathscr{G}) = M \otimes_A N$. By Theorem 14.3.D in Vakil, we only need to check that for any distinguised $\text{Spec } A_f \hookrightarrow \text{Spec } A$, the map $\Gamma(\text{Spec } A, \mathscr{F} \otimes \mathscr{G})_f \to \Gamma(\text{Spec } A_f, \mathscr{F} \otimes \mathscr{G})$ is an isomorphism. This amounts to showing $(M \otimes_A N)_f \cong M_f \otimes_{A_f} N_f$. We will show this by using universal properties. Consider the diagram

$$\begin{tikzcd}
M \times N \arrow[r] \arrow[d] & M \otimes_A N \arrow[r] & (M \otimes_A N)_f \\
M_f \times N_f \arrow[d] & & \\
M_f \otimes_{A_f} N_f \arrow[uurr, leftrightarrow] & &
\end{tikzcd}$$

Since the vertical map $M \times N \to M_f \otimes_{A_f} N_f$ is A-bilinear, there is a map $M \otimes_A N \to M_f \otimes_{A_f} N_f$, and in this map f becomes invertible in $M_f \otimes_{A_f} N_f$, and so there is a map $\phi : (M \otimes_A N)_f \to M_f \otimes_{A_f} N_f$ of A_f-modules. On the other hand, the horizontal map $M \times N \to (M \otimes_A N)_f$ makes f invertible in $(M \otimes_A N)_f$, so there exist a map $(M \times N)_f \cong M_f \times N_f \to (M \otimes_A N)_f$, and this map is A_f-bilinear, and so we have a map of A_f-modules $\psi : M_f \otimes_{A_f} N_f \to (M \otimes_A N)_f$. The map ϕ and ψ are inverses of each other, and hence $(M \otimes_A N)_f \cong M_f \otimes_{A_f} N_f$. Thus $\Gamma(\text{Spec } A, \mathscr{F} \otimes \mathscr{G})_f \to \Gamma(\text{Spec } A_f, \mathscr{F} \otimes \mathscr{G})$ is an isomorphism, so $\mathscr{F} \otimes \mathscr{G}$ is quasicoherent.

Question 3.

Suppose $\mathscr{F}$ is a quasicoherent sheaf. We define $\text{Sym}^n \mathscr{F}$ to be the sheafification of the presheaf $\text{Sym}^n \mathscr{F} : U \mapsto \text{Sym}^n(\mathscr{F}(U))$. Using the algebraic fact (given B is an A-algebra) $\text{Sym}^k_A(M) \otimes_A B \cong \text{Sym}^k_B(M \otimes_A B)$ and applying it to $B = A_f$ we see that $(\text{Sym}^k_A)_f \cong \text{Sym}^k_{A_f}(M_f)$. Thus given $U = \text{Spec } A$ an affine open of X where $\Gamma(\text{Spec } A, \mathscr{F}) = M$, we see that $(\text{Sym}^k_A)_f = \Gamma(\text{Spec } A, \text{Sym}^k \mathscr{F})_f \cong \Gamma(\text{Spec } A_f, \text{Sym}^k \mathscr{F}) = \text{Sym}^k_{A_f}(M_f)$, and so by Theorem 14.3.D in Vakil, $\text{Sym}^k \mathscr{F}$ is quasicoherent.

Similarly, if $\mathscr{F}$ is a quasicoherent sheaf, we define $\bigwedge^n \mathscr{F}$ to be the sheafification of the presheaf $\bigwedge^n \mathscr{F} : U \mapsto \bigwedge^n(\mathscr{F}(U))$. Again we use the algebraic fact that $\bigwedge^k_A(M) \otimes_A B \cong \bigwedge^k_B(M \otimes_A B)$ which we apply to $B = A_f$ to conclude $(\bigwedge^k_A)_f \cong \bigwedge^k_{A_f}(M_f)$. Thus another application of 14.3.D shows that $\bigwedge^k \mathscr{F}$ is quasicoherent.

Suppose $\mathscr{F}$ is locally free of rank m, then from our construction, let $U = \text{Spec } A$ be an affine open where $\mathscr{F}|_{\text{Spec } A} = \tilde{A}^{\oplus m}$, then our comstruction shows that $\Gamma(\text{Spec } A, \text{Sym}^k \mathscr{F}) = \text{Sym}^k(\tilde{A}^{\oplus m})$ which is free of rank $\begin{pmatrix} k+m-1 \\ m-1 \end{pmatrix}$, and $\Gamma(\text{Spec } A, \bigwedge^k \mathscr{F}) = \bigwedge^k(\tilde{A}^{\oplus m})$ is free of rank $\begin{pmatrix} m \\ k \end{pmatrix}$.

Question 4.

We consider a small enough open set Spec A where $\mathscr{F}, \mathscr{F}', \mathscr{F}''$ are free. Suppose $\Gamma(\text{Spec } A, \mathscr{F}) = \tilde{A}^{\oplus p+q}, \Gamma(\text{Spec } A, \mathscr{F}') = \tilde{A}^{\oplus p}$, and $\Gamma(\text{Spec } A, \mathscr{F}'') = \tilde{A}^{\oplus q}$, then using the algebra fact $\text{Sym}(M \oplus N) \cong \text{Sym}M \otimes \text{Sym}N$, we see that

$$\text{Sym}^k \mathscr{F}|_{\text{Spec } A} \cong \oplus_{i=0}^k \text{Sym}^i \mathscr{F}'|_{\text{Spec } A} \otimes_A \text{Sym}^{k-i} \mathscr{F}''|_{\text{Spec } A}$$

Define $\mathscr{F}^p = \oplus_{i=p}^k \text{Sym}^i \mathscr{F}'|_{\text{Spec } A} \otimes_A \text{Sym}^{k-i} \mathscr{F}''|_{\text{Spec } A}$. This is a well defined subsheaf because on the level of open sets U, $\mathscr{F}^p(U) = \oplus_{i=p}^k \text{Sym}^i \mathscr{F}'|_{\text{Spec } A}(U) \otimes_A \text{Sym}^{k-i} \mathscr{F}''|_{\text{Spec } A}(U) \hookrightarrow \oplus_{i=0}^k \text{Sym}^i \mathscr{F}'|_{\text{Spec } A}(U) \otimes_A \text{Sym}^{k-i} \mathscr{F}''|_{\text{Spec } A}(U)$ is an injection as they are direct sum of modules, so using 3.4.N in Vakil states that $\mathscr{F}^p$ are well defined subsheaf of $\text{Sym}^k \mathscr{F}|_{\text{Spec } A}$. So we see that there is a filtration of subsheafs

$$\text{Sym}^k \mathscr{F}|_{\text{Spec } A} = \mathscr{F}^0 \supset \mathscr{F}^1 \supset \ldots \supset \mathscr{F}^r \supset \mathscr{F}^{r+1} = 0$$

Now by our construction it follows that $\mathscr{F}^p/\mathscr{F}^{p+1} \cong \mathrm{Sym}^p\mathscr{F}|_{\mathrm{Spec}\ A} \otimes \mathrm{Sym}^{k-p}\mathscr{F}|_{\mathrm{Spec}\ A}$ as they are all direct sums of free sheaves and so the quotient is just the summand in $\mathscr{F}^p$ that is not in $\mathscr{F}^{p+1}$, which is $\mathrm{Sym}^p\mathscr{F}|_{\mathrm{Spec}\ A} \otimes \mathrm{Sym}^{k-p}\mathscr{F}|_{\mathrm{Spec}\ A}$.

Question 5.

Suppose $\mathscr{F}$ is coherent and $\mathscr{G}$ is quasicoherent. We will consider the Hom sheaf $\mathcal{H}om(\mathscr{F},\mathscr{G})$ on affine opens. On affine open $U = \mathrm{Spec}\ A$, let $\Gamma(\mathrm{Spec}\ A, \mathscr{F}) = M$ and $\Gamma(\mathrm{Spec}\ A, \mathscr{G}) = N$, then $\Gamma(\mathrm{Spec}\ A, \mathcal{H}om(\mathscr{F},\mathscr{G})) = \mathrm{Hom}(\Gamma(\mathrm{Spec}\ A, \mathscr{F}), \Gamma(\mathrm{Spec}\ A, \mathscr{G})) = \mathrm{Hom}(M, N)$. We will again use Theorem 14.3.D in Vakil to show quasicoherent. Since $\mathscr{F}$ is coherent, M is finitely presented, and hence by Theorem 2.6.G in Vakil, which states that $S^{-1}\mathrm{Hom}_A(M,N) \cong \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$, we see that $\mathrm{Hom}_A(M,N)_f \cong \mathrm{Hom}_{A_f}(M_f, N_f)$ (by applying 2.6.G to f), which is precisely $\Gamma(\mathrm{Spec}\ A, \mathcal{H}om(\mathscr{F},\mathscr{G}))_f \cong \Gamma(\mathrm{Spec}\ A_f, \mathcal{H}om(\mathscr{F},\mathscr{G}))$, and thus by 14.3.D $\mathcal{H}om(\mathscr{F},\mathscr{G})$ is quasicoherent.

Question 6.

Suppose $\mathscr{F}$ is a quasicoherent sheaf of finite type on a scheme X, we want to show that the support is closed. Since $\mathscr{F}$ is quasicoherent, we again reduce the case to when X is affine. Given A a ring, M an A-module, we let $X = \mathrm{Spec}\ A$ and $\mathscr{F} = \widetilde{M}$. First of all, given a section s, it is true that in any sheaf the support of a section is closed. We will show that for quasicoherent sheaf it is of the form $V(ann(s))$ where ann is the annihilator. Suppose $p \in \mathrm{Spec}\ A$ and suppose $a \in ann(s)$ is such that $a \in A - p$. Now $a \in ann(s)$ means $as = 0$, which implies that s_p, the image of s in M_p, is zero. On the other hand, if $s_p = 0$, then by definition of localization there exist $a \in A - p$ such that $as = 0$, and hence there is an element $a \in ann(s) \cap (A - p)$. Thus we have shown that

$$\{p \in \mathrm{Spec}\ A : s_p = 0\} = \{p \in \mathrm{Spec}\ A : ann(s) \text{ is not contained in } p\}$$

and hence the complement of the above two sets, $supp(s)$ and $V(ann(s))$, are equal.

Now consider $supp(\mathscr{F})$. Since M is finitely generated A-module, we let $m_1, ..., m_n$ be generators, then $supp(\mathscr{F}) = \bigcup_s supp(s)$. However, as M is finitely generated by $m_1, ..., m_n$, we see that $supp(\mathscr{F}) = \bigcup_{i=1}^n supp(m_i) = \bigcup_{i=1}^n V(ann(m_i)) = V(\prod_{i=1}^n ann(m_i)) = V(ann(M))$ which gives a nice expression of $supp(\mathscr{F})$ as a closed subset.

Finally, we will show that the support of a quasicoherent sheaf need not be closed. Let $A = \mathbb{C}[t]$, then $\mathbb{C}[t]/(t-a)$ is an A-module supported at a. Consider the case $\mathscr{F} = \widetilde{M}$ where $M = \oplus_{a\in\mathbb{C}}\mathbb{C}[t]/(t-a)$, this $\widetilde{M}$ is quasicoherent by definition, but the support is not closed: for the prime ideals $(t-a_0)$ for any $a_0 \in \mathbb{C}$, we see that $(\oplus_{a\in\mathbb{C}}\mathbb{C}[t]/(t-a)) \otimes \mathbb{C}[t]_{(t-a_0)} = \oplus_{a\in\mathbb{C}}(\mathbb{C}[t]/(t-a) \otimes \mathbb{C}[t]_{(t-a_0)})$ is not zero because the a_0-th summand is not zero, and hence $\mathscr{F}$ is support for all $(t-a_0) \in \mathrm{Spec}\ \mathbb{C}[t]$. However, consider $(0) \in \mathrm{Spec}\ \mathbb{C}[t]$, we see that $(\oplus_{a\in\mathbb{C}}\mathbb{C}[t]/(t-a)) \otimes \mathbb{C}(t) = \oplus_{a\in\mathbb{C}}(\mathbb{C}[t]/(t-a) \otimes \mathbb{C}(t)) = 0$ because all summand are 0. Thus $supp(\mathscr{F}) = \mathrm{Spec}\ \mathbb{C}[t] - (0)$, which is not closed.