Math 256A Problem Set 8

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Question 1.

Let $X =$ Spec $k[x_1, x_2, ...]$, and let $U = X - V(m)$ where m is the maximal ideal $(x_1, x_2, ...)$. We will take two copies of X (denote by X_1 and X_2), and we will glue them together along the open set U_1 and U_2 where $U_i = X_i - V(m)$ for $i = 1, 2$. We glue them together by just using the identity isomorphism of schemes $(U_1, \mathcal{O}_{X_1}|_{U_1}) \overset{i}{\leftrightarrow} (U_2, \mathcal{O}_{X_2}|_{U_2})$. We denote the resulting space by W, and we will show that this is not quasiseparated. Consider the two affine opens $X_1, X_2 \subset W$, then $X_1 \cap X_2 \cong U_1 \cong U_2$, and 4.6.G in Vakil showed that U_1 (or U_2) is not quasicompact, and hence it cannot possibly be a finite union of affine open subsets (since finite union of affine opens are quasicompact). Thus W is not quasiseparated.

Question 2.

Let B be a finite A-algebra, define a graded ring $S_\bullet$ by $S_0 = A$ and $S_n = B$ for all $n > 0$. Given any prime ideal p in B, we let p' be a homogeneous prime ideal in $S_\bullet$ where $(p')_0 = A$ and $(p')_n = p$ for all $n > 0$. This is homogeneous (generated by homogeneous elements as $S_n = B$ for all $n > 0$), and it is prime because if $a' \in S_n$ and $b' \in S_m$ are such that $a'b' \in (p')_{n+m}$, then when considered as just elements of p, we see that $a' \in p$ or $b' \in p$. Thus $a' \in (p')_n$ or $b' \in (p')_m$, and hence p' is a homogeneous prime ideal. On the other hand, given a homogeneous prime ideal p', then we will show that $(p')_n = (p')_m$ for all $n, m > 0$. Suppose $b \in (p')_n$ for some n, then for $1 \in S_1$, we see that $1 \cdot b = b \in (p')_{n+1}$, thus $b \in (p')_m$ for all $m > n$. On the other hand, let $1 \in S_1$ and $b \in S_{n-1}$, then $1 \cdot b = b \in (p')_n$, so either $1 \in (p')_1$ or $b \in (p')_{n-1}$. However, $1 \notin (p')_1$ so we see that $b \in (p')_{n-1}$. Thus we see that $b \in (p')_m$ for all $0 < m < n$. Thus for p' a homogeneous prime ideal, we send it to $p'' = (p')_1$, which is a prime ideal in B because $(p')_n = (p')_m$ for all m, n. From our construction, the map $p \in SpecB \mapsto p' \in ProjS_\bullet \mapsto p'' = p \in SpecB$ is the identity, and $p' \in ProjS_\bullet \mapsto p'' \in SpecB \mapsto p' \in ProjS_\bullet$ is also the identity. Thus we see that we have a bijection of points in the topological space $f : ProjS_\bullet \sim SpecB$. Now this map f takes $D_+(h)$ to $D(h)$ because if h is contained in a homogeneous prime ideal p', then $h \in (p')_n$ for all n, and hence it will be in p'' and vice versa. Therefore we have a homeomorphism $ProjS_\bullet \cong SpecB$. It gives an isomorphism on the structure sheaf because given any $h \in S_n$, we see that $B_h \cong ((S_\bullet)_h)_0$ (since $S_n = B$), and thus we have that $ProjS_\bullet \cong SpecB$ is an isomorphism.

Question 3.

Suppose $X =$ Spec A and $Y =$ Spec B, and π : Spec $A \to$ Spec B, then given any point $q \in$ Spec B, we know from 10.3 in Vakil that the fiber above q is Spec $(A \otimes_B B_q/qB_q)$. We will show that the map π'' : Spec $(A \otimes_B B_q/qB_q) \to$ Spec (B_q/qB_q) is a finite morphism, and we will do this in two steps. We can choose to localize first and then take quotient, or we can quotient first and then localize, but it doesn't matter as the square below commutes (5.3.4.1 in Vakil):

$$\begin{array}{ccc} B & \xrightarrow{\textit{localize}} & B_q \\ {\scriptstyle quotient}\downarrow & & {\scriptstyle quotient}\downarrow \\ B/q & \xrightarrow{\textit{localize}} & B_q/qB_q = K(B/q) \end{array}$$

So we will quotient first. By assumption, $\pi^\sharp : B \to A$ makes A a finite B-algebra. Consider the following corresponding diagrams

$$\begin{array}{ccc} \text{Spec }(A \otimes_B B/q) & \longrightarrow & \text{Spec } A \\ \pi' \downarrow & & \pi \downarrow \\ \text{Spec } B/q & \longrightarrow & \text{Spec } B \end{array} \qquad \begin{array}{ccc} A \otimes_B B/q & \longleftarrow & A \\ (\pi')^\sharp \uparrow & & \pi^\sharp \uparrow \\ B/q & \longleftarrow & B \end{array}$$

Since $\pi^\sharp : B \to A$ makes A a finite B-algebra, we let $a_1, ..., a_n$ be the generators for A as a B-module, then we see that $A \otimes_B B/q$ is finitely generated as a B/q-module by $a_1 \otimes \bar{1}, ..., a_n \otimes \bar{1}$ because

$$\begin{aligned} (\pi^\sharp(b_1)a_1 + ... + \pi^\sharp(b_n)a_n) \otimes \bar{b} &= \pi^\sharp(b_1)a_1 \otimes \bar{b} + ... + \pi^\sharp(b_n)a_n \otimes \bar{b} \\ &= a_1 \otimes \bar{b_1}\bar{b} + ... + a_n \otimes \bar{b_n}\bar{b} \\ &= (\pi')^\sharp(b_1 b)(a_1 \otimes \bar{1}) + ... + (\pi')^\sharp(b_n b)(a_n \otimes \bar{1}) \end{aligned}$$

Thus we see that $(\pi')^\sharp : B/q \to A \otimes_B B/q$ makes $A \otimes_B B/q$ into a finite B/q-algebra.

On the other hand, we will show that localization also does not change the finiteness condition. Again we consider the following corresponding diagrams

$$\begin{array}{ccc} \text{Spec }(A \otimes_B B_q) & \longrightarrow & \text{Spec } A \\ \pi' \downarrow & & \pi \downarrow \\ \text{Spec } B_q & \longrightarrow & \text{Spec } B \end{array} \qquad \begin{array}{ccc} A \otimes_B B_q & \longleftarrow & A \\ (\pi')^\sharp \uparrow & & \pi^\sharp \uparrow \\ B_q & \longleftarrow & B \end{array}$$

Following an almost identical computation, we see that $(\pi')^\sharp : B_q \to A \otimes_B B_q$ makes $A \otimes_B B_q$ into a finite B_q-algebra.

Now as the fiber above q is given by the square of fibers

$$\begin{array}{ccc} \text{Spec }(A \otimes_B B_q/qB_q) & \longrightarrow & \text{Spec } A \\ \pi'' \downarrow & & \pi \downarrow \\ \text{Spec } B_q/qB_q & \longrightarrow & \text{Spec } B \end{array}$$

and π'' can be obtained by first taking quotient and localizing, we see that π'' is finite, and as B_q/qB_q is a field, by 8.3.H, the fiber Spec $A \otimes B_q/qB_q$ is finite. Thus we conclude that finite morphisms have finite fibers.

Question 4.

(a) For open immersions, we can just consider $(U, \mathcal{O}_X|_U) \overset{i}{\hookrightarrow} (X, \mathcal{O}_X)$ an inclusion of open subscheme $(U, \mathcal{O}_X|_U)$. Suppose Spec B is an affine open in X, then $i^{-1}(\text{Spec } B) = U \cap \text{Spec } B$ is open in X and in Spec

B, so we let $D(f) \in i^{-1}(\text{Spec } B)$ be an basic open where $f \in B$. Then we see that the induced morphism is just the map to localization $B \to B_f$, and B_f is a finitely generated B-algebra (generated by 1). Thus we see that the inclusion $(U, \mathcal{O}_X|_U) \stackrel{i}{\hookrightarrow} (X, \mathcal{O}_X)$ is locally of finite type. Thus we can conclude that open immersions are locally of finite type, and hence quasicompact open immersion is of finite type.

We will show that every open immersion into a locally Noetherian scheme is of finite type. From 8.1.B (or by part (c)), if $f : X \to Y$ is an open immersion, then if Y is a locally Noetherian ring, then so is X. Now 8.3.B show that every morphism from a locally Noetherian scheme is quasicompact, and thus as open immersions are locally of finite type, these two combined inplies that f is of finite type. Thus every open immersion into a locally Noetherian scheme is of finite type.

(b) Consider $f : X \to Y$ and $g : Y \to Z$ each locally of finite type. Then given any affine open Spec$B \subset$, we see that $g^{-1}(\text{Spec } B)$ can be covered by affine opens U_i such that the induced map $B \to U_i$ makes U_i into a finitely generated B-module. On the other hand, for each of these U_i, $f^{-1}(U_i)$ can be covered by affine open V_{ij} where the induced map $U_i \to V_{ij}$ makes V_{ij} into a finitely generated U_i-module. Then we see that $(g \circ f)^{-1}\text{Spec } B = \bigcup V_{ij}$ is a union of affine opens, and $B \to V_{ij}$ makes V_{ij} a finitely generated B-module (as it is the composition $B \to U_i \to V_{ij}$). Thus we conclude by 8.3.O that the composition $g \circ f$ is locally of finite type.

(c) Suppose $f : X \to Y$ is locally of finite type, and Y is locally Noetherian. Then we see that Y can be covered by affine open Spec B_i where B_i is Noetherian for all $i \in I$ some indexing set. Now by definition of morphisms locally of finity type, we see that any affine open subset A_{ij} (here $j \in J$ an indexing set dependant on what i is) of $f^{-1}(\text{Spec } B_i)$ the induced morphism $B_i \to A_{ij}$ makes A_{ij} a finitely generated B_i-algebra. However, every finitely-generated commutative algebra over a commutative Noetherian ring is Noetherian, and thus A_{ij} are Noetherian for all i, j. Now as the B_i covers Y, we see that A_{ij} covers X, and thus X is locally Noetherian. If $f : X \to Y$ is of finite type, and Y is Noetherian, then the quasicompactness of f implies only finitely many affine open sets A_{ij} (we get these A_{ij} following an almost identical argument as before) cover X, and hence X is Noetherian.

Question 5.

Vakil 3.5 stated that exactness of a sequence of sheaves may be checked at the level of stalks. Suppose $\mathcal{G}' \to \mathcal{G} \to \mathcal{G}''$ is an exact sequence of $\mathcal{O}_X$-modules, then we see that $\mathcal{G}'_x \to \mathcal{G}_x \to \mathcal{G}''_x$ is exact for all $x \in X$. Now by right exactness of tensor product of $\mathcal{O}_{X,x}$-modules, we see that $\mathcal{G}'_x \otimes_{\mathcal{O}_{X,x}} \mathcal{F}_x \to \mathcal{G}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{F}_x \to \mathcal{G}''_x \otimes_{\mathcal{O}_{X,x}} \mathcal{F}_x$ is exact. Now as $(\mathcal{G}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{F}_x) \cong (\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F})_x$ we see that $(\mathcal{G}' \otimes_{\mathcal{O}_X} \mathcal{F})_x \to (\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F})_x \to (\mathcal{G}'' \otimes_{\mathcal{O}_X} \mathcal{F})_x$ is exact. Thus we see that $(\mathcal{G}' \otimes_{\mathcal{O}_X} \mathcal{F}) \to (\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}) \to (\mathcal{G}'' \otimes_{\mathcal{O}_X} \mathcal{F})$ is exact on stalks, and hence it is exact. (Here I did not use locally free sheaf, and it seems to also be right exact. I do not know what is missing).

Question 6.

We will check the axiom of groups. Tensor product is associative, and for commutative rings R and R-algebras A, B, there is an isomorphism $A \otimes_R B \cong B \otimes_R A$, and thus we see that for sheaves of $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$, there is an isomorphism $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \cong \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}$. Given any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, we see that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong \mathcal{F}$, and 14.1.D shows that if $\mathcal{F}$ is an invertible sheaf, then $\mathcal{F} \otimes \mathcal{F}^\vee \cong \mathcal{O}_X$. Thus we see that the invertible sheaves on X up to isomorphism form an abelian group with identity $\mathcal{O}_X$, and for every element $\mathcal{F}$ the inverse is $\mathcal{F}^\vee$.