Math 256A Problem Set 7

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Question 1.

1. Given any two closed immersions $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$ of X and any affine open $U = SpecB \subset X$, we have $f_1^{-1}(U) \cong Spec(B/I_1)$ and $f_2^{-1}(U) \cong Spec(B/I_2)$ for some I_1, I_2. In question 9.1,G, we defined the intersection of f_1 and f_2 to be a closed immersion $f : Y \to X$ such that at $f^{-1}(U)$ it is $Spec(B/(I_1 + I_2))$. We will interpret this as a fibered product. On the affine open B, consider the following diagram

$$\begin{array}{ccc} Spec(B/I_1) \times_B Spec(B/I_2) & \longrightarrow & Spec(B/I_1) \\ \downarrow & & \downarrow \\ Spec(B/I_2) & \longrightarrow & Spec(B) \end{array}$$

we see that $Spec(B/I_1) \times_B Spec(B/I_2) = Spec(B/I_1 \otimes_B B/I_2) = Spec(B/\langle I_1, I_2\rangle) = Spec(B/(I_1+I_2))$, which concide with what we defined in 9.1.G. Now since in both constructions we construct the scheme on affine opens and glue them together, and the two contructions agree on affine opens, we see that the intersection of two closed immersions into X is the fibered product over X.

2. In 10.1.1 in Vakil he showed that open immersions are preserved by base change, and in 10.2.1 he showed that closed immersions are preserved by base change. Suppose $f : X \to Z$ is locally closed immersion, we factor the map f into $X \overset{g}{\to} Y \overset{h}{\to} Z$ where g is a closed immersion and h is an open immersion, then using this diagram

$$\begin{array}{ccc} W \times_Z X & \longrightarrow & X \\ g' \downarrow & & g \downarrow \\ W \times_Z Y & \longrightarrow & Y \\ h' \downarrow & & h \downarrow \\ W & \longrightarrow & Z \end{array}$$

we see that h' is an open immersion and g' is a closed immersion by 10.1.1 and 10.2.1, so we see that $W \times_Z X \to Z$ is a locally closed immersion. Therefore locally closed immersions are preserved by base change.

3. We can define the intersection of finite number of locally closed immersions in X (denote by $f_i : Y_i \to X$ for $i \in \{1, 2, ..., n\}$) to be the finite fibered product $Y_1 \times_X Y_2 \times_X ... \times_X Y_n$.

Question 2.

Since in the construction of fibered products we take fibered products on affine opens and glue them together, we will show that for $X = SpecA$ and $Y = SpecB$ locally finite type k-scheme, $X \times_k Y$ is also locally finite type over k. Now as $X \times_k Y = Spec(A \otimes_k B)$, and A and B are finitely generated k-algebra (say they are generated by $a_1, ..., a_n$ and $b_1, ..., b_m$ respectively), then we see that $a_i \otimes b_j$ (where $i \in \{1, ..., b\}$ and $j \in \{1, ..., m\}$) generates $A \otimes_k B$ as k-algebras (since it generates $A \otimes_k B$ as k-vector space). This shows that $X \times_k Y = Spec(A \otimes_k B)$ is locally of finite type k-scheme. Therefore we see that for general X and Y locally finite type k-scheme, $X \times_k Y$ is also locally finite type over k. Now if X and Y are quasicompact, then $X \times_k Y$ is quasicompact, so if X and Y are finite type k-schemes, $X \times_k Y$ is also of finite type over k.

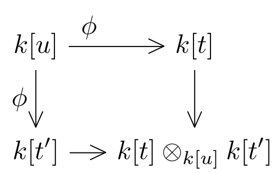
Question 3.

First we write down what $\mathbb{A}^2_k \times \mathbb{P}^1_k$ is. On the affine open $D_+(u) = Spec(k[u,v]_u)_0 \cong Spec[v/u]$, we see that the fibered product is $Spec(k[x,y] \otimes k[v/u])$, and similarly on $D_+(v)$ the fibered product is $Spec(k[x,y] \otimes k[u/v])$, and we glue them together along their intersection. Since $Bl_{(0,0)}\mathbb{A}^2_k$ is cut out by the equation $xv = yu$, we see that it is glued together by $Spec(k[x,y,\frac{v}{u}]/(x\frac{v}{u} - y))$ and $Spec(k[x,y,\frac{u}{v}]/(x - \frac{u}{v}y))$. Now without lost of generality we can suppose a closed point p is in $Spec[v/u]$, then it is of the form $(v/u - c)$ for some $c \in k$, and its fiber is $Spec(k[\frac{v}{u}]/(\frac{v}{u} - c) \otimes_{k[\frac{v}{u}]} k[x,y,\frac{v}{u}]/(x\frac{v}{u} - y)) = Spec(k[x,y,\frac{v}{u}]/(x\frac{v}{u} - y, \frac{v}{u} - c)) = Spec(k[x,y]/(cx - y))$ so the fiber is just cut out by a line of slope c.

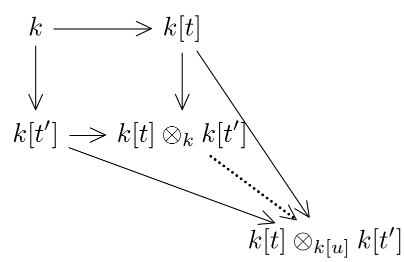
On the other hand, on $D_+(u)$ the fiber above (x,y) is $Spec(k[x,y,\frac{v}{u}]/(x\frac{v}{u} - y) \otimes_{k[x,y]} k[x,y]/(x,y)) = Spec([k[x,y,\frac{v}{u}]/(x\frac{v}{u} - y)]/(x,y)) = Spec([k[x,y,\frac{v}{u}]/(x\frac{v}{u} - y)]/(x)) = Spec(R/(x))$ if we let $R = k[x,y,\frac{v}{u}]/(x\frac{v}{u} - y)$. So we see that it is locally principal because locally it is cut out by one equation (x), and it is not locally a zerodivisor because x is not a zerodivisor in R. To briefly describe what the fiber looks like, we see that the above quotient actually simplifies to $Spec(k[\frac{v}{u}])$, and similarly on $D_+(v)$ it is $Spec(k[\frac{u}{v}])$, and they are glued by $\frac{v}{u} \to \frac{u}{v}$, so we recovered the whole $\mathbb{P}^1_k$.

Question 4.

Let $\phi : k[u] \to k[t]$ by $u \mapsto t^2$. Consider the pushout of rings



The pushout $k[t] \otimes_{k[u]} k[t']$ is actually a $k[u]$-algebra, so it is automatically a k-algebra, and the diagram above is actually a k-algebra diagram. Thus the above diagram actually fits in the following diagram (of rings and also of k-algebras):



where the square is just the obvious maps, and $k[t] \otimes_k k[t']$ is the pushout for that square. Now as $k[t] \otimes_k k[t']$ satisfies the universal property of pushouts, we see that there exist a map $k[t] \otimes_k k[t'] \to k[t] \otimes_{k[u]} k[t']$. The map sends $a \otimes b \mapsto a \otimes b$, but in $k[t] \otimes_{k[u]} k[t']$, we have $t^2 \otimes 1 = 1 \otimes (t')^2$, and hence $t^2 \otimes 1 - 1 \otimes (t')^2 \mapsto 0$. Thus we see that $k[t] \otimes_k k[t']/(t^2 \otimes 1 - 1 \otimes (t')^2) \cong k[t] \otimes_{k[u]} k[t']$. But $k[t] \otimes_k k[t']/(t^2 \otimes 1 - 1 \otimes (t')^2) \cong k[t,t']/(t^2 - (t')^2)$, we see that $k[t] \otimes_{k[u]} k[t'] \cong k[t,t']/(t^2 - (t')^2) \cong k[t,t']/((t^2 - t')(t + t')) \cong k[u,v]/(uv)$ which has two irreducible component by a question we did in one previous homework.