Math 256A Problem Set 6

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Question 1.

Suppose X is a quasicompact scheme (so write as $X = \cup_{i=1}^{n} U_i$ for U_i affine), then given any point $p \in X$, we can let $p \in U_1$ (renumber if needed) be an affine open containing p. Then as U_1 is affine, we see that the closure of p contains a closed point p_1 of U_1 (since $p \in U_1$ must be contained in a maximal ideal) If p_1 is closed in X then it is good. Otherwise p_1 is not closed in X, then we take the closure of p_1, and there must be a point p_2 that is not p_1. Since $p_2 \neq p_1$, we see that $p_2 \notin U_1$, so we let (renumber if needed) $p_2 \in U_2$. Now as the closure of p_2 inside U_2 contains a closed point (U_2 is affine), we can pick p_2 to be the closed point in U_2. If p_2 is closed in X, then we are done. Otherwise take the closure again of p_2, and take p_3 in the closure. Now as $p_3 \in \overline{\{p_2\}} \subset \overline{\{p_1\}}$, we see that $p_3 \notin U_1 \cup U_2$, and hence by a similar argument can pick $p_3 \in U_3$ to be a maximal ideal in U_3. Continuing this way we see that as X is quasicompact, this process must terminate (as we have only finitely many U_i), and hence the closure of p in X must contain a closed point. It follows that every non-empty closed subset of X contains a closed point of X, and in particular, every non-empty quasicompact schemes has a closed point.

Question 2.

Suppose a scheme X is quasicompact, then X can be covered by a finite number of affine opens because a scheme can always be covered by an arbitrary amount of affine opens. If X is also quasiseparated, then we see that the intersection of any two such can be covered by finitely many affine opens.

On the other hand, suppose X can be covered by a finite number of affine opens, then since affine open schemes are quasicompact, a finite union of them are also quasicompact. If the intersection of any two such can be covered by finitely many affine opens, I was unable to show that the intersection of any two affine open is the union of affine open subsets.

Question 3.

Suppose a scheme X is integral, then by definition, X is reduced because integral domains have no non-zero nilpotents. X is irreducible because if not, then there exist two open subset U_1, U_2 that does not intersect. But then $O_X(U_1 \cup U_2)$ is just the product $O_X(U_1) \times O_X(U_2)$, which is not an integral domain, a contradiction. Thus X is irreducible.

On the other hand, if X is reduced and irreducible, we let U be any affine open. Then we see that U is irreducible. Now as $U = SpecA$ is irreducible, there is a unique minimal prime ideal p_0 in A. Since $O_X(U)$ has no nonzero nilpotent, we see that $p_0 \subset \cap p = nil(A) = (0)$, and so (0) is prime, and hence $O_X(U)$ is an integral domain. Now given any arbitrary open set V, we consider the injective map $O_X(V) \to \prod_{x \in V} O_{X,x}$. Note that $O_{X,x}$ is an integral domain (as sections on affine opens are integral domains). Given any $f, g \in O_X(V)$ such that $fg = 0$, we let $\{f_x\}$, $\{g_x\}$ be the images of f, g in $\prod_{x \in V} O_{X,x}$ (so $f_x g_x = 0 \in O_{X,x}$). The sets $A = \{x \in V : f_x = 0\}$ and $B = \{x \in V : g_x = 0\}$ are closed and $A \cup B = V$, so $A = V$ or $B = V$ as V is irreducible. Thus we see that f or g is 0, and hence $O_X(V)$ is an integral domain. As V is arbitrary, we see that X is integral.

Question 4.

1. We will denote $\mathbb{Z}/(x^2-n) = \mathbb{Z}[\sqrt{n}]$ and $K(\mathbb{Z}/(x^2-n)) = \mathbb{Q}/(x^2-n) = \mathbb{Q}[\sqrt{n}]$ where $K(-)$ is taking the field of fractions. We will use a theorem in algebraic number theory that states that the integral closure of a ring R in an extension S is integrally closed. We let $S = \mathbb{Q}[\sqrt{n}]$ and $R = \mathbb{Z}$. Now suppose $a + b\sqrt{n} \in \mathbb{Q}[\sqrt{n}]$ is integral over $\mathbb{Z}$, we note first of all that $a - b\sqrt{n}$ is also integral, and as integral elements form a ring, it follows that $2a$ and $a^2 - nb^2$ are also integral over $\mathbb{Z}$. So we see that $2a$ and $a^2 - nb^2$ are in $\mathbb{Z}$. Note then also $4nb^2 \in \mathbb{Z}$. If $a \notin \mathbb{Z}$, then we see that $n \equiv 1 \bmod 4$, which is not possible, and hence $a \in \mathbb{Z}$. Then $nb^2 \in \mathbb{Z}$, and so together with $4nb^2 \in \mathbb{Z}$ shows that $4b^2$ is even, and thus $2b$ is even and hence $b \in \mathbb{Z}$. Thus $a + b\sqrt{n} \in \mathbb{Z}[\sqrt{n}]$. Thus we see that $\mathbb{Z}/(x^2-n)$ is integrally closed, and hence $Spec(\mathbb{Z}/(x^2-n))$ is normal.

2. We will use question 6.4.H to do this part. 6.4.H states that if A is a unique factorization domain with 2 invertible, $f \in A$ has no repeated prime factors, and $z^2 - f$ is irreducible in $A[z]$, then $A[z]/(z^2-f)$ is normal. We let $A = k[x_1, ..., x_{m-1}, x_{m+1}, x_n]$, which is a UFD, and we let $f = -(x_1^2 + ...x_{m-1}^2)$, which has no repeated prime factors (being irreducible) and $x_m^2 + x_1^2 + ...x_{m-1}^2$ is irreducible in $A[x_n]$. Thus question 6.4.H helps us conclude that $A[x_m]/(x_m^2 - f) = k[x_1, ..., x_n]/(x_1^2 + ... + x_m^2)$ is normal.

3. We will use 6.4.J to do reduce it to the case of part (b). Question 6.4.J states that all quadratic form in n variable can, under change of coordinates, be expressed as a sum of at most n squares where the elements in the squares are linearly independent. Applying this, we see that $wz - xy = ((w+z)/2)^2 + (i(w-z)/2)^2 + (i(x+y)/2)^2 + ((x-y)/2)^2$, and hence by the previous part of this problem, we see that $k[w,x,y,z]/(wz-xy)$ is normal.

Question 5.

Given x a homogeneous element in S_+, we define a map $D(f(x)) \to D(x)$ by $p \mapsto f^{-1}(p)$. This makes sense because if p is generated by homogeneous elements $\{r_i\}_{i \in I}$, then $f^{-1}(p)$ is generated by the homogeneous elements $\{f^{-1}(r_i)\}_{i \in I}$. The map of schemes corresponding to this is the map $Spec(((S_\bullet)_x)_0) \to Spec(((R_\bullet)_{f(x)})_0)$. Thus we have construct maps $\pi_j : D(f(x_j)) \to Proj S_\bullet$. We want to show that these morphisms glue. Given any fixed x_i, x_j such that $D(f(x_i))$ and $D(f(x_j))$ intersect, we consider the sheafs maps $\pi_i|_{Spec(((S_\bullet)_{x_i x_j})_0)}$ and $\pi_j|_{Spec(((S_\bullet)_{x_i x_j})_0)}$. We see that for i this corresponds to the restriction of π_i to $D(x_j/x_i)$ and at j this corresponds to the restriction of π_j to $D(x_i/x_j)$ (corresponding to sections $(((s_\bullet)_{x_i})_0)_{x_j/x_i}$ and $(((s_\bullet)_{x_j})_0)_{x_i/x_j}$). However, we see that the image of this morphism of sheaves is $(((R_\bullet)_f(x_i))_0)_{f(x_j)/f(x_i)}$ under π_i and $(((R_\bullet)_f(x_j))_0)_{f(x_i)/f(x_j)}$, but these two rings are glued together in $Proj(R_\bullet)$, and hence we see that π_i and π_j agree on the intersections. Thus we have defined a map $Proj(R_\bullet) - V(f(S_+)) \to Proj(S_\bullet)$.

Question 6.

Suppose $S_\bullet$ is a finitely generated ring which is generated in degree 1, then we let $a_1, ..., a_m \in S_1$ be the generators. Now as $a_1, ..., a_m$ generates $S_\bullet$, we see that S_n consists of finite sums of elements of the form $ca_1^{i_1}...a_m^{i_m}$ where $c \in S_0$ and $i_1 + ... + i_m = n$ for all n. Thus fixing a n now, we see that S_{nj} for any integer j consists of finite sums of elements of the form $a_1^{i_1}...a_m^{i_m}$ where $i_1 + ... + i_m = nj$, but $a_1^{i_1}...a_m^{i_m}$ can be expressed as products $a_1^{i_1}...a_m^{i_m} = (a_1^{(i_1)_1}...a_m^{(i_m)_1})(a_1^{(i_1)_2}...a_m^{(i_m)_2})...(a_1^{(i_1)_j}...a_m^{(i_m)_j})$ where $(i_1)_k + ... + (i_m)_k = n$ for all $k \in \{1, ..., j\}$. Thus we see that $S_{n\bullet} = \oplus_{j=0}^{\infty} S_{nj}$ is generated in S_n, and hence $S_{n\bullet}$ is also generated in degree 1.

Question 7.

If $S_\bullet$ and $R_\bullet$ are the same finitely generated graded rings except in a finite number of nonzero degrees, i.e. if $S_{n\bullet} \cong R_{n\bullet}$ for all but a finite amount of integers n, and we pick n_0 to be the largest of the n such that they are not isomorphic. Then by question 7.4.D, which states $Proj S_\bullet \cong Proj S_{n\bullet}$, we see that for $n > n_0$, $Proj S_\bullet \cong Proj S_{n\bullet} \cong Proj R_{n\bullet} \cong Proj R_\bullet$.