Math 256A Problem Set 5

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Question 1.

We will proceed by showing that any prime ideal in a ring R (that is not the zero ring) contains at least one minimal prime ideal. Given any prime ideal $\mathfrak{p} \subset R$, we let S be the set of prime ideals in R contained in $\mathfrak{p}$. The set S is not empty as $\mathfrak{p} \in S$. Now consider any chain of primes $... \subset \mathfrak{p}_i \subset \mathfrak{p}_{i+1} \subset ...$ in S, we see that $\cap_i \mathfrak{p}_i$ is again a prime ideal (intersection of chain of prime ideals is a prime ideal), and it is the lower bound of the chain. Thus every chain in S has a lower bound in S, so Zorn's lemma tells us that there is at least one minimal element in S, so $\mathfrak{p}$ contains at least one minimal prime ideal.

Now suppose $SpecA$ is irreducible, then we know by the argument above that there is at least one minimal prime ideal. We will show that there is exactly one. Suppose $\{p_i\}_{i \in I}$ are the set of all minimal prime ideals in A, where I is some indexing set, and suppose I can be expressed as a union of non-empty J and K where $J \cap K$ is empty. Then the two sets $D(\cup_{j \in J} p_j)$ and $D(\cup_{k \in K} p_k)$ are non-empty, open, and disjoint (as all prime ideals contain at least one minimal prime ideal). This shows that $SpecA$ is reducible, a contradiction. Thus I cannot be expressed as a union of non-empty J and K where $J \cap K$ is empty, which means I must have only one element (I is not empty). Therefore if $SpecA$ is irreducible, there is exactly one minimal prime ideal.

On the other hand, suppose there is exactly one minimal prime ideal $\mathfrak{p}_0$, then the first paragraph imples that all prime ideal contains $\mathfrak{p}_0$. Thus all non-empty basis open in $SpecA$ contains $\mathfrak{p}_0$, so any two non-empty open sets have non-empty intersection, which means that $SpecA$ is irreducible.

Question 2.

The prime ideals in the ring $R/I = k[x,y]/(xy)$ correspond to prime ideals in $k[x,y]$ that contain (xy), which are prime ideals in $k[x,y]$ containing x or y. Now the prime ideals containing x correspond to primes in $k[x,y]/(x) \cong k[y]$ and similarly prime ideals containing y correspond to primes in $k[x,y]/(y) \cong k[x]$. The prime ideal containing both x and y (and hence also xy) is the maximal ideal (x,y). Thus $\text{Spec } k[x,y]/(xy) \cong \text{Spec } k[x] \cup_{(0,0)} \text{Spec } k[y]$ where the two "lines" are identified at (x,y). The minimal prime ideals in $k[x,y]/(xy)$ can be found as follows: if a prime $\mathfrak{p}$ contains x, then its image in $k[x,y]/(x) \cong k[y]$ contains the unique minimal prime ideal $(0) \subset k[y]$, which, under the isomorphism, is $(x) \subset k[x,y]$. Similarly, if a minimal prime $\mathfrak{p}$ contains y, then its image in $k[x,y]/(y) \cong k[x]$ contains the unique minimal prime ideal $(0) \subset k[x]$, which, under the isomorphism, is $(y) \subset k[x,y]$. Thus there are precisely two minimal prime ideals in $k[x,y]/(xy)$: (x) and (y).

Question 3.

Consider the ring given in the previous exercise: $A = k[x,y]/(xy)$. Spec A is reducible, because the two open sets $D(x) = \text{Spec } (k[x,y]/(xy))_x \cong \text{Spec } k[x]_x$ and $D(y) = \text{Spec } (k[x,y]/(xy))_y \cong \text{Spec } k[y]_y$ are disjoint non-empty open subsets of Spec A (their union is not the whole Spec A though).

On the other hand, suppose $\text{Spec } A = X \coprod Y$ where X and Y are disjoint non-empty closed subset. Note that in Spec A, $\overline{\{(x)\}} = V((x))$ and $\overline{\{(y)\}} = V((y))$, where $V(x) \cup V(y) = \text{Spec } A$ and $V(x) \cap V(y) = (x,y)$.

Thus if $(x), (y)$ are both in X, then Y is empty, a contradiction. On the other hand, if $(x) \in X$ and $(y) \in Y$, then X, Y are not disjoint, a contradiction. Thus $\text{Spec } A$ is connected.

Question 4.

Suppose A is Noetherian. First note that connected components are closed (this is true in any topological space) because for any connected set, its closure is also connected. Let $A = \cup_{i \in I} C_i$ where C_i are connected components and I is an indexing set. By Proposition 4.6.6 proved in Vakil, every connected components (since they are closed sets) can be uniquely expressed as a finite union $C_i = Z_{1i} \cup ... \cup Z_{ni}$ of irreducible closed subsets, non contained in any other. Now as the connected components are disjoint, we see that Z_{ij} is not contained in Z_{kl} for all $j \neq l$. Thus these Z_{ij} are not contained in any larger irreducible sets, and hence they are precisely the irreducible components. Thus connected components are unions of the irreducible components.

Now apply again Proposition 4.6.6 in Vakil to the Noetherian topological space $\text{Spec } A$, we see that it can be uniquely expressed as a finite union $Z_1 \cup ... \cup Z_n$ of irreducible closed subsets, non contained in any other. Now as irreducible sets are automatically connected, we see that the number of connected components cannot be more than n, and hence is finite. Since connected components are disjoint closed sets that cover the entire topological space $SpecA$, finiteness of the number of connected components implies that every connected component is both open and closed. On the other hand, if a set S is open and closed, it cannot be properly contained in any connected component. Therefore S must be the unions of connected components of $\text{Spec}A$.

Question 5.

Suppose A is a Noetherian ring and M is a finitely generated A-module, then we will show that M is a Noetherian module. First of all, if $0 \to M' \to M \to M'' \to 0$ is exact, then M is Noetherian if and only if M' and M'' are Noetherian (4.6.V in Vakil). Since M is finitely generated A-module, we let $S = \{f_1, ..., f_n\}$ be the set of generators of M. Then we know that $F(S)$, the free module generated by the set S, satisfies the property that $F(S) \to M$ by mapping $f_i \mapsto f_i$ is a module homomorphism onto M. Now as $F(S) \cong A^{\oplus n}$ and $A^{\oplus n}$ is a Noetherian A-module (4.6.W in Vakil), we see that we have an exact sequence

$$0 \to \ker \phi \to A^{\oplus n} \xrightarrow{\phi} M \to 0$$

where the middle term $A^{\oplus n}$ is Noetherian. Thus M is a Noetherian A-module.

Question 6.

I was unable to come up with a proof. Fix n, I was thinking that since the functor sends a ring R to rank 1 free submodules in R^{n+1}, it kind of resembles the construction of projective space of $\mathbb{R}^{n+1}$, which are lines through origin. So I tried to let $X = Proj\mathbb{Z}[x_0, ..., x_n]$, so then for any rank 1 free submodules in R^{n+1} generated by $(r_0, ..., r_n)$, consider a ring homomorphism $\phi : \mathbb{Z}[x_0, ..., x_n] \to R$ by mapping $1 \mapsto 1$ and $x_i \mapsto r_i$, so a prime $p \in SpecR$ is mapped to a prime $\phi^{-1}(p)$ in $\mathbb{Z}[x_0, ..., x_n]$ which is homogeneous. So we have a map $SpecR \to Proj\mathbb{Z}[x_0, ..., x_n]$ as a topological space. However, I have not showed that this is a bijection of sets

$$\{ \text{ free rank 1 submodule L of } R^{n+1} \text{ such that there is L' with } L \oplus L' = R^{n+1}\} \leftrightarrow Hom(SpecR, X)$$

and I did not even show that the thing I though about gave a morphism of schemes. Thus I do not have any proof, just some ideas.

Question 7.

(a) We will construct an isomorphism of schemes $\text{Spec } A_1 \coprod \text{ Spec } A_2 \coprod ... \coprod \text{ Spec } A_n \to \text{ Spec } A$, where $A = A_1 \times A_2 \times ... \times A_n$, thereby showing that a disjoint union of a finite number of affine schemes is an affine scheme. First of all we note that all prime ideal of A are of the form $A_1 \times ... \times A_{i-1} \times P_i \times A_{i+1} \times ... \times A_n$ where P_i is a prime in A_i. We define $\phi : \text{ Spec } A_1 \coprod \text{ Spec } A_2 \coprod ... \coprod \text{ Spec } A_n \to \text{ Spec } A$ by mapping $p_i \mapsto A_1 \times ... \times A_{i-1} \times p_i \times A_{i+1} \times ... \times A_n$. We first check that this ϕ is a homeomorphism of the topological space. On each of the $\text{Spec} A_i$, the map ϕ maps $D(a_i)$ bijectively onto $D(f_i)$ where $f_i = (0, ..., 0, a_i, 0, ..., 0)$. This shows that ϕ is not only a continuous bijection, but actually an open mapping, and hence a homeomorphism.

We will now construct a morphism of structure sheaves of rings (on a base), and show that it is an isomorphism. Note first that if a base element $D(r) \subset SpecA$ is such that more than 1 component of r are non-zero, then $D(r) = \emptyset$, so we can take the base of $SpecA$ to be $D(r)$ where r is of the form $(0, ..., 0, r_i, 0, ..., 0)$. Now we define $\psi : \mathcal{O}_{SpecA} \to \phi_* \mathcal{O}_{\text{ Spec } A_1 \coprod ... \coprod \text{ Spec } A_n}$ by letting

$$\begin{tikzcd}
\psi(D(r)) \colon \mathcal{O}_{SpecA}(D(r)) \arrow[r] \arrow[d, "="'] & \phi_* \mathcal{O}_{\text{ Spec } A_1 \coprod ... \coprod \text{ Spec } A_n}(D(r)) = \mathcal{O}_{\text{ Spec } A_1 \coprod ... \coprod \text{ Spec } A_n}(\phi^{-1} D(r)) \arrow[d, "="'] \\
A_{(0,...,0,r_i,0,...,0)} & (A_i)_{r_i}
\end{tikzcd}$$

be the map $A_{(0,...,0,r_i,0,...,0)} \to (A_i)_{r_i}$ sending $(a_1, a_2, ..., a_i, ..., a_n)/(0, ..., 0, (r_i)^k, 0, ...0) \mapsto a_i/(r_i)^k$. This is an isomorphism which commutes with restriction maps, and so ψ is an isomorphism of structure sheaf. Therefore there is an isomorphism of schemes $\text{Spec } A_1 \coprod \text{ Spec } A_2 \coprod ... \coprod \text{ Spec } A_n \to \text{ Spec } A$, and hence a finite disjoint union of affine schemes is again affine.

(b) Infinite disjoint union of affine schemes is never an affine scheme, as affine schemes are quasi-compact.