Math 256A Problem Set 4

Kuan-Ying Fang

September 30, 2011

Question 1.

(a) The projection map $\phi : B \to B/I$ induces a bijection between the set Spec B/I and the prime ideals in B that contains I (this was proved in the last homework). However, the set of prime ideals in B containing I is by definition precisely the set $V(I)$, thus we see that Spec B/I is naturally a closed subset of Spec B.

If $S = \{1, f, f^2, ...\}$, then the inclusion $\psi : B \to S^{-1}B$ induces a bijection between Spec $S^{-1}B$ and the prime ideals in B not intersecting S. However, the prime ideals in B not intersecting S is precisely the set $\{p \in \text{Spec } B : f \notin S\} = D(f)$, so Spec $S^{-1}B$ is naturally an open subset of Spec B.

However, when S is arbitrary, then Spec$S^{-1}B$ need not be open or closed. An example is Spec$\mathbb{C}[x, y]_{(x,y)}$. This, when considered as a subset of Spec$B$, is neither closed nor open. To show this, suppose Spec$\mathbb{C}[x, y]_{(x,y)}$ is closed, then it is equal to $V(W)$ for some $W \subset \mathbb{C}[x, y]$. However, as Spec$\mathbb{C}[x, y]_{(x,y)}$ consists of prime ideals contained in (x, y), we see that $(0) \in$ Spec$\mathbb{C}[x, y]_{(x,y)}$, and hence $W \subset (0)$, which means that $W = \{0\}$, but this can not be right because then $V(W) = \text{Spec}\mathbb{C}[x, y] \neq \text{Spec}\mathbb{C}[x, y]_{(x,y)}$, and thus Spec$\mathbb{C}[x, y]_{(x,y)}$ is not closed. On the other hand, suppose it is open, then it is equal to some $D(S) = \{p \in \text{Spec}\mathbb{C}[x, y] : S \text{ is not contained in } p\}$. However, as (x, y) is maximal, if there is some S not contained in (x, y), then $S + (x, y)$ is another ideal properly containing (x, y), contradicting the maximality. Thus no such S exists, and hence Spec$\mathbb{C}[x, y]_{(x,y)}$ is not open. Thus Spec$\mathbb{C}[x, y]_{(x,y)}$ is neither open nor closed.

(b) The Zariski topology on Spec B/I states that the closed sets are of the form $V(S)$ for some $S \subset B/I$. $V(S)$ corresponds, under the bijection induced by $\phi : B \to B/I$, to the set $\{p \in \text{Spec } B : I \subset p, \phi^{-1}(S) \subset p\} = V(I) \cap V(\phi^{-1}(S)) = \text{Spec}B/I \cap V(\phi^{-1}(S))$. Thus the Zariski topology on SpecB/I is the subspace topology induced by inclusion in Spec$B$.

Now we consider $S^{-1}B$. The Zariski topology on Spec$S^{-1}B$ defines closed sets to be of the form $V(W)$ for some $W \subset S^{-1}B$. $V(W)$ corresponds, under the bijection induced by $\phi : B \to S^{-1}B$, to the set $\{p \in \text{Spec } B : p \cap S = \emptyset, \phi^{-1}(W) \subset p\} = \text{Spec}S^{-1}B \cap V(\phi^{-1}(W))$. Thus the Zariski topology on Spec$S^{-1}B$ is the subspace topology induced by inclusion in Spec$B$.

Question 2.

Suppose A is an integral domain, then given any two non-empty open sets, we want to show that they have nonempty intersection. Now as $\{D(f)\}_{f \in A}$ form a base for the topological space Spec A, we see that any two open sets U, V can be written as $U = \cup_i D(f_i)$ and $V = \cup_j D(g_j)$. Since A is an integral domain, (0) is prime, and as $D(f) = \{p \in \text{Spec}A : f \notin p\}$, we see that $(0) \in D(f)$ for all $f \neq 0$ (if f is zero, then the set $D(f)$ is empty, so it does not really matter). Thus we see that $(0) inU = \cup_i D(f_i)$ and $(0) \in V = \cup_j D(g_j)$, and thus $(0) \in U \cap V$, which shows that the intersection is nonempty. Thus we have shown that if A is an integral domain, Spec$A$ is irreducible.

Question 3.

Suppose that $p \in \text{Spec}A$ is maximal, then we see that $V(p) = \{p' \in \text{Spec}A : p \subset p'\} = \{p\}$ by maximality.

Thus $\{p\}$ is closed. On the other hand, suppose $\{p\}$ is a closed point, then by the definition of the Zariski topology, we see that $\{p\} = V(I)$ for some ideal I. Since we are assuming the ring A has 1, every ideal is contained in a maximal ideal, so let $p' \supset I$ be such maximal ideal. Then as $\{p\} = V(I)$, we see that $p' = p$ and hence p is maximal.

Question 4.

(a) Suppose Spec $A = \cup_i U_i$, then since $\{D(f)\}_{f\in A}$ form a base, we see that Spec $A = \cup_i U_i = \cup_{i,j} D(f_{i,j})$ where $U_i = \cup_j D(f_j)$. Then since Spec $A = \cup_{i,j} D(f_{i,j})$, we see that $(f_{i,j}) = A$, and so $1 \in (f_{i,j})$. This means that there exists some $a_{i,j}$, all but finitely many 0 (denote the (i,j) where $a_{i,j}$ is not zero by S'), such that $1 = \sum_{i,j} a_{i,j} f_{i,j}$, but this means that $(f_{i,j})_{S'} = A$, so we see that $\cup_{i,j\in S'} D(f_{i,j})$. Since S' is finite, we let I' be the i such that i,j is in S', then I' is finite, and we see that Spec $A = \cup_{i\in I'} U_i$, and thus Spec$A$ is quasicompact.

(b) Consider $A = k[x_1, x_2, x_3, ...]$ and $\mathfrak{m} = (x_1, x_2, x_3, ...) \subset A$. Then we will show that the set $V(\mathfrak{m})^c = \{p \in SpecA : p \text{ is not contained in } m\}$ is not compact. First we see that $V(\mathfrak{m})^c = \{p \in SpecA : m \text{ is not contained in } p\} = \cup_{i\in\mathbb{N}} D(x_i)$. However, let I be any finite subset of the natural numbers $\mathbb{N}$, then we see that $\cup_{i\in I} D(x_i)$ does not contain the prime $(x_i)_{i\in J}$ for any $J \supset I$. But $(x_i)_{i\in J}$ is in $V(\mathfrak{m})^c$, so we see that the cover $V(\mathfrak{m})^c = \cup_{i\in\mathbb{N}} D(x_i)$ does not have a finite subcover. Thus it is not quasicompact. This show that Spec$A$ can have nonquasicompact open subsets.

Question 5.

Suppose $\{p\}$ is a specialization of $\{q\}$, then $\{p\} \in \overline{\{q\}}$. We see by definition of closure that

$$\overline{\{q\}} = \cap_{q\in V(S)} V(S)$$

where the intersection is taken over all S such that $V(S)$ contains q. Now if $\{p\} \in \overline{\{q\}}$, then for all S such that $S \subset q$, we have $S \subset p$. However, $q \subset q$, so we see that $q \subset p$. On the other hand, if $q \subset p$, then it follows trivially that for all S such that $S \subset q$, we have $S \subset p$, and hence $\{p\} \in \overline{\{q\}}$.

Using the above statement, we see that if $\{q\} \in V(p)$, then $p \subset q$, so $\{q\} \in \overline{\{p\}}$, while if $\{q\} \in \overline{\{p\}}$, then $p \subset q$ and so $\{q\} \in V(p)$. Thus $V(p) = \overline{\{p\}}$.

Question 6.

(a) Define a map $\phi : k[w,x,y,z] \to k[a,b]$ by taking $w \mapsto b^3$, $x \mapsto ab^2$, $y \mapsto a^2b$, and $z \mapsto a^3$. We want to show that $I = (wz - xy, wy - x^2, xz - y^2)$ is equal to $\ker(\phi)$. The inclusion $I \subset \ker(\phi)$ is clear by just plugging in values. We will show that $I \supset \ker(\phi)$. We will proceed by constructing basis for $k[w,x,y,z]$ as a vector space over k. For degree n, we see that the polynomials consisting of element of only degree n is a $\binom{n+4-1}{n}$-dimensional vector space, with the "conventional" basis $B_n = \{w^{a_1}x^{a_2}y^{a_3}z^{a_4} : a_1 + a_1 + a_3 + a_4 = n\}$. We let $S_n = \{w^{a_1}x^{a_2}y^{a_3}z^{a_4} : a_2 + a_3 \leq 1 \text{ and } a_1 + a_1 + a_3 + a_4 = n\}$. Then we see that all elements of $B_n - S_n$ can be written as a sum of an element from S and an element from I. (Justification: for any $w^{a_1}x^{a_2}y^{a_3}z^{a_4}$ such that $a_2 + a_3 \geq 2$, we can reduce the number $a_2 + a_3$ by writing it as $w^{a_1}x^{a_2}y^{a_3}z^{a_4} = w^{a_1}x^{a_2-1}y^{a_3-1}z^{a_4}(xy - wz) + w^{a_1+1}x^{a_2-1}y^{a_3-1}z^{a_4+1}$ or $w^{a_1}x^{a_2}y^{a_3}z^{a_4} = w^{a_1}x^{a_2-2}y^{a_3}z^{a_4}(x^2 - wy) + w^{a_1+1}x^{a_2-2}y^{a_3+1}z^{a_4+1}$ or $w^{a_1}x^{a_2}y^{a_3}z^{a_4} = w^{a_1}x^{a_2}y^{a_3-2}z^{a_4}(y^2 - xz) + w^{a_1}x^{a_2+1}y^{a_3-2}z^{a_4+1}$ depending on what the a_1, a_2 actually are. Thus we can reduce the sum $a_1 + a_2$ in each step, which justifies the statement that all elements of $B_n - S_n$ can be written as a sum of an element from S_n and an element from I.) Thus we see that we can pick a basis for degree n polynomials consisting of elements in S_m for $m \leq n$ and elements in I. Note that there are $3n + 1$ elements in S_n, and they map bijectively under ϕ to elements $\{a^{r_1}b^{r_2} : r_1 + r_2 = 3n\}$. Now suppose $p(w,x,y,z)$ is a degree n polynomial in $k[w,x,y,z]$ such that $p(w,x,y,z) \in ker\phi$, then we first write $p(x) = s(x) + i(x)$ where $s(x)$ are described as above (sum of elements in S_m for $m \leq n$), and $i(x) \subset I$, then we see that $\phi(p(x)) = \phi(s(x)) = 0$. However, this cannot be possible if $s(x) \neq 0$, as we have just stated that elements of S_n map bijectively under ϕ to elements $\{a^{r_1}b^{r_2} : r_1 + r_2 = 3n\}$, and thus by counting the degrees in $k[a, b]$ we see that $s(x) = 0$, and hence $p(x) \subset I$. Now by the first isomorphism theorem, we see that $k[w, x, y, z]/I = k[w, x, y, z]/Ker\phi \cong Im\phi$ where $Im\phi$ is the subring generated by monomials of degree divisible by 3. Thus $k[w, x, y, z]/I$ is an integral domain, and hence I is prime and Spec$k[w, x, y, z]/I$ is irreducible by Question 2 above.

(b) The generators of the ideal of part (a) can be written as the equations ensuring that

$$\text{rank} \begin{pmatrix} w & x & y \\ x & y & z \end{pmatrix} \leq 1$$

since for the above matrix to have rank less than or equal to one means that all 2×2 minor of this matrix is zero, which is the condition in I. Now we consider an ideal I in $k[x_0, x_1, ..., x_n]$ such that the generators are the equations ensuring that the $2 \times n$ matrix satisfies

$$\text{rank} \begin{pmatrix} x_0 & x_1 & x_2 & ... & x_{n-1} \\ x_1 & x_2 & x_3 & ... & x_n \end{pmatrix} \leq 1$$

then we can again apply similar strategy to show that Spec$k[x_0, x_1, ..., x_n]/I$ is irreducible by showing that I is prime. We will map $\phi : k[x_0, x_1, ..., x_n] \to k[a, b]$ by mapping $x_i \to a^i b^{n-i}$. Then we see that $ker\phi \supset I$ as any 2×2 minor, i.e. $x_{i_1}x_{i_2} - x_{i_1+1}x_{i_2-1}$, is mapped to $a^{i_1}b^{n-i_1}a^{i_2}b^{n-i_2} - a^{i_1+1}b^{n-i_1-1}a^{i_2-1}b^{n-i_2+1} = 0$. To show that $ker\phi \subset I$, we again define $S_m = \{x_0^{a_0}x_1^{a_1}...x_m^{a_m} : a_1 + a_2 + ... + a_{m-1} \leq 1, \text{ and } a_0 + a_1 + a_2 + ... + a_m = m\}$, then we see that polynomials of degree m can be written as a sum of an element in I and elements in S_l for $l \leq m$. Again, we see that S_m is mapped bijectively by ϕ onto the set $\{a^{r_1}b^{r_2} : r_1 + r_2 = mn\}$, and thus we see that (by a similar argument of counting degrees in $k[a, b]$) $I = ker\phi$, and hence by the first isomorphism theorem, we see that $k[x_0, x_1, ..., x_n]/I = k[x_0, x_1, ..., x_n]/Ker\phi \cong Im\phi$ where $Im\phi$ is the subring generated by monomials of degree divisible by n. Thus $k[x_0, x_1, ..., x_n]/I$ is an integral domain, and hence I is prime and Spec$k[x_0, x_1, ..., x_n]/I$ is irreducible by Question 2 above.