Math 256A Problem Set 3

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Question 1.

(a) We will verify that a morphism of sheaves is determined by the induced morphism of sheaves on the base. We let $\phi_1, \phi_2 : \mathscr{F} \to \mathscr{G}$ be two morphism of sheaves, and denote $\overline{\phi_1}, \overline{\phi_2} : F \to G$ to be the induced morphism of sheaves on the base. Then the claim that we wish to verify is the same as saying that if for any given open $U = \bigcup_i B_i \subset X$ where B_i are base elements, and suppose $\overline{\phi_1}(B_i) = \overline{\phi_2}(B_i)$ for all i, then $\phi_1 = \phi_2$.

To show this, we first note that given any base element B', the induced morphism $\overline{\phi_1}(B')$ is defined to be $\phi_1(B')$ (and similarly for ϕ_2). Given any open $U = \bigcup_i B_i \subset X$ (where B_i are base elements) such that $\overline{\phi_1}(B_i) = \overline{\phi_2}(B_i)$ for all i, we will treat these morphisms as elements of the "Sheaf Hom". First of all, $\overline{\phi_1}(B_i) = \phi_1(B_i) \in \mathcal{H}om(\mathscr{F}, \mathscr{G})(B_i)$, and as we have constructed in the last homework, $\overline{\phi_1}(B_i) = \phi_1(B_i) = res_{U,B_i}(\phi_1(U))$ where $\phi_1(U) \in \mathcal{H}om(\mathscr{F}, \mathscr{G})(U)$. Thus as $\overline{\phi_1}(B_i) = res_{U,B_i}(\phi_1(U))$ for all i (and similarly for ϕ_2), we have

$$res_{U,B_i}(\phi_1(U)) = \overline{\phi_1}(B_i) = \overline{\phi_2}(B_i) = res_{U,B_i}(\phi_2(U))$$

So as "Sheaf Hom" is a sheaf, by the identity axiom on the Sheaf Hom, we see that $\phi_1(U) = \phi_2(U)$, and as U is arbitrary, we conclude that $\phi_1 = \phi_2$.

(b) Given F, G two sheaves on the base, we can construct, as described in this section, sheaves $\mathscr{F}, \mathscr{G}$ on X. Given $\overline{\phi} : F \to G$ a morphism of sheaves on the base and any open $U \subset X$, we want to define a map $\phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$. We will do so by considering compatible stalks. By definition

$$\mathscr{F}(U) = \{(f_p \in F_p)_{p \in U} : \text{ for all } p \in U, \text{ there exists B with } p \subset B \subset U, s \in F(B), \text{ with } s_q = f_q \text{ for all } q \in B\}$$

However, given any p we see that $\overline{\phi} : F \to G$ induces a morphism of stalks $\overline{\phi}_p : F_p \to G_p$. So first of all we consider a set

$$S = \{(\overline{\phi}_p(f_p) \in G_p)_{p \in U} : (f_p) \in \mathscr{F}(U)\}$$

We want to show that this is contained in $\mathscr{G}(U)$. To see this, given $p \in U$ there exists B with $p \in B \subset U, s \in F(B)$ with $s_q = f_q$ for all $q \in B$. Then given any $p \in U$, we take the same $p \in B \subset U$, and consider $\overline{\phi}(B)(s) \in G(B)$, we see by the commutative square (as described in Section 3.4.3.1)

$$\begin{array}{ccc} F(B) & \xrightarrow{\overline{\phi}(B)} & G(B) \\ \downarrow & & \downarrow \\ \prod_{q \in B} F_q & \xrightarrow{\prod_{q \in B} \overline{\phi}_q} & \prod_{q \in B} G_q \end{array}$$

that if $s_q = f_q$ for all $q \in B$, then

$$\overline{\phi}(B)(s)_q = \overline{\phi}_q(s_q) = \overline{\phi}_q(f_q)$$

Thus $\overline{\phi}(B)(s)$ satisfies $\overline{\phi}(B)(s)_q = \overline{\phi}_q(f_q)$ for all $q \in B$, and hence we see that S is a subset of $\mathscr{G}(U)$, and thus by sending $f_p \mapsto \overline{\phi}_p(f_p)$, we have constructed a map $\phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ for an arbitrary U, and hence we have a data of maps $\phi : \mathscr{F} \to \mathscr{G}$. The tautological restriction map commutes with the data of maps that we have just contructed, and thus we conclude that $\phi : \mathscr{F} \to \mathscr{G}$ as defined is a morphism of induced sheaves, and a morphism of sheaves on the base gives a morphism of the induced sheaves.

Question 2.

Since $X = \bigcup_i U_i$, then we define $B_{i,j}$, $j \in J_i$ where J_i is an indexing set depending on i, be open sets contained in U_i, then $\{B_{i,j}\}$ form a base of X. Now for any $B_{i,j}$, we can choose k (depending on i, j) such that $B_{i,j} \subset U_k$ and $F(B_{i,j}) = \mathscr{F}_k(B_{i,j})$ (potentially using the axiom of choice?). Then given any $B_1 \subset B_2$ with the corresponding k_1, k_2, we define the restriction map by $\text{res}_{B_2,B_1} = \phi_{k_2k_1} \circ res_{B_2,B_1}$ where the second restriction map is the restriction in $\mathscr{F}_{k_2}$ (the restriction of sheaves will be italicize, while the restriction of sheaves on base will not be). We will check that this restriction map gives a presheaf on base. Suppose $B_1 \subset B_2 \subset B_3$ with corresponding k_1, k_2, k_3, then

$$\begin{aligned}
\text{res}_{B_2,B_1} \circ \text{res}_{B_3,B_2} &= \phi_{k_2k_1} \circ res_{B_2,B_1} \circ \phi_{k_3k_2} \circ res_{B_3,B_2} \\
&= \phi_{k_2k_1} \circ \phi_{k_3k_2} \circ res_{B_2,B_1} \circ res_{B_3,B_2} \\
&= \phi_{k_3k_1} \circ res_{B_3,B_1} = \text{res}_{B_3,B_1}
\end{aligned}$$

where the second equality comes from $\phi_{k_3k_2}$ being an isomorphism of sheaves, so it commutes with restriction maps. Thus we have a presheaf on base.

We will check that this defines a sheaf on base. We will check base identity first. Given $B = \bigcup_i B_i$, with corresponding k_i for B_i and k for B. Then if $f, g \in F(B)$ are such that $\text{res}_{B,B_i} f = \text{res}_{B,B_i} g$ for all i, then we see that $\phi_{kk_i} \circ res_{B,B_i} f = \phi_{kk_i} \circ res_{B,B_i} g$ for all i, and thus $res_{B,B_i} f = res_{B,B_i} g$ for all i. Thus $f = g$ by the sheaf identity axiom on $\mathscr{F}_k$.

Next we check base gluability. Suppose $B = \bigcup_i B_i$, with corresponding k_i for B_i and k for B. Suppose we have $f_i \in F(B_i)$ for all i such that $\text{res}_{B_i,B_i \cap B_j} f_i = \text{res}_{B_j,B_i \cap B_j} f_j$ (here we let k_{ij} correspond to $B_i \cap B_j$), then we see that $\phi_{k_ik_{ij}} \circ res_{B_i,B_i \cap B_j} f_i = \phi_{k_jk_{ij}} \circ res_{B_j,B_i \cap B_j} f_j$, so $\phi_{k_{ij}k} \circ \phi_{k_ik_{ij}} \circ res_{B_i,B_i \cap B_j} f_i = \phi_{k_{ij}k} \circ \phi_{k_jk_{ij}} \circ res_{B_j,B_i \cap B_j} f_j$, and hence $\phi_{k_ik} \circ res_{B_i,B_i \cap B_j} f_i = \phi_{k_jk} \circ res_{B_j,B_i \cap B_j} f_j$. Therefore, we see that $res_{B_i,B_i \cap B_j} \circ \phi_{k_ik} f_i = res_{B_j,B_i \cap B_j} \circ \phi_{k_jk} f_j$ for all i, j, and hence by the gluability of the sheaf $\mathscr{F}_k$, we see that there is an element $f \in \mathscr{F}_k(B) = F(B)$ such that $res_{B,B_i} f = \phi_{k_ik} f_i$ for all i, but then $\text{res}_{B,B_i} f = \phi_{kk_i} \circ res_{B,B_i} f = \phi_{kk_i} \circ \phi_{k_ik} f_i = f_i$ for all i, so we see that $f \in F(B)$ is the element satisfying the base gluability. Thus F is base gluable, and we see that F is a sheaf on base.

Finally, since we are given a sheaf on base, we can construct a $\mathscr{F}$ on X which is unique up to unique isomorphism by Theorem 3.7.1 of Vakil. It also claims that $F(B) \cong \mathscr{F}(B)$, and thus we see that given $B \subset U_i$, $\mathscr{F}_i(B) \cong F(B) \cong \mathscr{F}(B) = \mathscr{F}|_{U_i}(B)$, thus $\mathscr{F}_i \cong \mathscr{F}|_{U_i}$.

Question 3.

$\mathbb{A}^1_{\mathbb{Q}} = \text{Spec } \mathbb{Q}[x]$. Now as this is a Euclidean domain, all ideals are principal. All prime ideals are thus of the form $(f(x))$ where $f(x)$ is an irreducible polynomial. I was unable to classify all irreducible polynomials over $\mathbb{Q}$. The picture will look like $\mathbb{C}$ where any given $c \in \mathbb{C}$ will be identified with all its Galois conjugate.

Question 4.

Suppose $\mathfrak{p}$ is a prime ideal, and suppose that it is not principal. Suppose on the contrary that for all $f, g \in \mathfrak{p}$, there is a common factor. Let $f(x, y)$ be a polynomial of smallest degree (of x and y combined).

Now as $\mathfrak{p}$ is not principal, we see that there is a $g(x, y)$ that is not a multiple of $f(x, y)$. But now f and g have a common factor, so $f(x, y) = f_1(x, y)f_2(x, y)$ where f_1 is a common factor of f and g. However, by primality, we see that either f_1 or f_2 is in $\mathfrak{p}$, which contradicts the fact that $f(x, y)$ is of the smallest degree. Thus the $f(x, y), g(x, y) \in \mathfrak{p}$ chosen here are such that f, g has no common factor.

Our choice of f, g here has no common factor in $\mathbb{C}[x, y] = \mathbb{C}[x][y]$. If f, when considered as a polynomial in $\mathbb{C}(x)[y]$, becomes a unit, then $f = x - a$ for some complex a (as f is of minimal degree). But then if $x - a \in \mathfrak{p}$, then given any $g(x, y) = r_n(x)y^n + r_{n-1}(x)y^{n-1} + ... + r_0(x)$, we see that $r_n(a)y^n + r_{n-1}(a)y^{n-1} + ... + r_0(a) \in \mathfrak{p}$, so some $y - b$ must be in $\mathfrak{p}$, and hence $\mathfrak{p} = (x - a, y - b)$. Otherwise, we see that f, g have no common factor in the Euclidean domain $\mathbb{C}(x)[y]$ by Gauss lemma applied to f. Then using the Eiclidean algorithm, we see that there exist $h_1, h_2 \in \mathbb{C}(x)[y]$ such that $fh_1 + gh_2$ is a unit in $\mathbb{C}(x)[y]$, which is of the form $h(x)$. Thus $fh_1 + gh_2 = h(x)$ We can clear the denominators of both sides, and we get $fh_1^{'} + gh_2^{'} = h'(x)$ where now $h_1^{'}, h_2^{'} \in \mathbb{C}[x, y]$ and $h'(x) \in \mathbb{C}[x]$. This gives $h'(x) \subset (f(x, y), g(x, y)) \subset \mathfrak{p}$, and thus by primality some $(x - a) \in \mathfrak{p}$, and by a similar argument some $(y - b) \in \mathfrak{p}$, and so $\mathfrak{p} = (x - a, y - b)$. This concludes the claim that the prime ideals of $\mathbb{C}[x, y]$ are of the form $(x - a, y - b)$ for some complex a, b, or a principal ideal $(f(x, y))$ generated by a irreducible polynomial $f(x, y)$.

Question 5.

Let A be a ring, $I \subset A$ an ideal, and $\phi : A \to A/I$ be the projection map. Given any prime $\mathfrak{p} \in \text{Spec}(A/I)$, we will first show that the map $\phi' : \text{Spec}(A/I) \to \{\mathfrak{p} \in \text{Spec}(A) : I \subset \mathfrak{p}\}$ defined by $\phi'(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$ is a prime containing I. Given any $a, b \in A$ such that $ab \in \phi'(\mathfrak{p})$, we see that $ab \in \phi^{-1}(\mathfrak{p})$, thus $\phi(ab) = \phi(a)\phi(b) \in \mathfrak{p}$, which means that $\phi(a)$ or $\phi(b)$ is in $\mathfrak{p}$. This shows that a or b is in $\phi^{-1}(\mathfrak{p})$, so $\phi'(\mathfrak{p})$ is a prime. It contains I because $0 \in \mathfrak{p}$ and thus $I = \phi^{-1}(0) \subset \phi^{-1}(\mathfrak{p})$, and hence $I \subset \phi'(\mathfrak{p})$.

On the other hand, given a prime $\mathfrak{p} \subset A$ containing I, we can consider the image of $\mathfrak{p}$ under ϕ, which is $\phi(\mathfrak{p}) = \mathfrak{p}/I$. Now given $(a + I), (b + I) \in A/I$ (where here I denote them by cosets), if $(a + I)(b + I) = (ab + I) \in \mathfrak{p}/I$, then we let $ab + I = p + I$ for some $p \in \mathfrak{p}$. We see that $ab - p \in I \subset \mathfrak{p}$, so $ab \in \mathfrak{p}$ and thus a or b is in $\mathfrak{p}$, and hence $(a + I)$ or $(b + I)$ is in $\mathfrak{p}/I$. Hence $\mathfrak{p}/I$ is a prime ideal in A/I. Thus there is a map $\phi'' : \{\mathfrak{p} \in \text{Spec}(A) : I \subset \mathfrak{p}\} \to \text{Spec}(A/I)$.

Now $\phi' \circ \phi''(\mathfrak{p}) = \phi'(\mathfrak{p}/I) = \phi^{-1}(\mathfrak{p}/I) = \mathfrak{p}$ and $\phi'' \circ \phi'(\mathfrak{p}) = \phi''(\phi^{-1}(\mathfrak{p})) = \phi(\phi^{-1}(\mathfrak{p})) = \mathfrak{p}$ as ϕ is surjective. Given $p_1 \subset p_2 \subset A/I$, it is clear that $\phi'(p_1) := \phi^{-1}(p_1) \subset \phi^{-1}(p_2) = \phi'(p_2)$. Thus ϕ' is a inclusion preserving bijection, with inverse ϕ''.

Question 6.

Let A be a ring and S a multiplicative set. Consider the map $\phi : A \to S^{-1}A$ by mapping $a \mapsto a/1$. We see by a similar argument as the previous question that there is a map $\phi' : \text{Spec}(S^{-1}A) \to \{\mathfrak{p} \in \text{Spec}(A) : \mathfrak{p} \cap S = \emptyset\}$ by mapping $\mathfrak{p} \in \text{Spec}(S^{-1}A)$ to $\phi^{-1}(\mathfrak{p}) \in \{\mathfrak{p} \in \text{Spec}(A) : \mathfrak{p} \cap S = \emptyset\}$. We will show that $\phi'(\mathfrak{p}) \cap S = \emptyset$. Suppose on the contrary that there is $s_0 \in \phi'(\mathfrak{p}) \cap S$, then $\phi(s_0) = s_0/1 \in \mathfrak{p}$. Now as $\mathfrak{p}$ is an ideal, we see that $(1/s_0)(s_0/1) = s_0/s_0 = 1/1$ is in $\mathfrak{p}$, and hence $\mathfrak{p}$ is the whole $S^{-1}A$, which contradicts the fact that $\mathfrak{p}$ is prime (hence proper).

On the other hand, we define a map $\phi'' : \{\mathfrak{p} \in \text{Spec}(A) : \mathfrak{p} \cap S = \emptyset\} \to \text{Spec}(S^{-1}A)$. Given any prime $\mathfrak{p} \in \text{Spec}(A)$ such that $\mathfrak{p} \cap S = \emptyset$, we let $\phi''(\mathfrak{p})$ be the subset $\mathfrak{p}'$ of $S^{-1}A$ consisting of elements of the form p_0/s_0 for some $p_0 \in \mathfrak{p}$. This set $\mathfrak{p}'$ is an ideal of $S^{-1}A$ as

$$\frac{p_0}{s_0} - \frac{p_1}{s_1} = \frac{p_0 s_1 - p_1 s_0}{s_0 s_1} \in \mathfrak{p}' \text{ and } \frac{p_0}{s_0} * \frac{p_1}{s_1} = \frac{p_0 p_1}{s_0 s_1} \in \mathfrak{p}'$$

It is prime because if a_0/b_0 and a_1/b_1 are any two elements of $S^{-1}A$ such that $(a_0 a_1)/(b_0 b_1) \in \mathfrak{p}'$, then

$$\frac{a_0 a_1}{b_0 b_1} = \frac{p_0}{s_0} \text{ for some } p_0 \in \mathfrak{p}, s_0 \in S$$

and hence there is some $s' \in S$ such that $s's_0a_0a_1 = s'p_0b_0b_1 \in \mathfrak{p}$, and as $\mathfrak{p} \cap S = \emptyset$, by the primality of $\mathfrak{p}$, we see that a_0 or a_1 is in $\mathfrak{p}$, thus a_0/b_0 or a_1/b_1 is in $\mathfrak{p}'$, and hence $\mathfrak{p}'$ is a prime ideal.

Now $\phi' \circ \phi''(\mathfrak{p}) = \phi'(\{a/b : a \in \mathfrak{p}, b \in S\}) = \phi^{-1}(\{a/b : a \in \mathfrak{p}, b \in S\}) = \mathfrak{p}$, On the other hand, if $a/b \in \mathfrak{p} \in Spec(S^{-1}A)$, then $a/1 \in \mathfrak{p}$, so $a \in \phi'(p)$, and thus $a/b \in \phi'' \circ \phi'(\mathfrak{p})$, while $\phi'' \circ \phi'(\mathfrak{p}) = \phi'' \circ \phi^{-1}(\mathfrak{p}) = \mathfrak{p}$, and thus ϕ' is a bijection. Given $p_1 \subset p_2 \subset S^{-1}A$, it is clear that $\phi'(p_1) := \phi^{-1}(p_1) \subset \phi^{-1}(p_2) = \phi'(p_2)$. Thus ϕ' is a inclusion preserving bijection, with inverse ϕ''.