Math 256A Problem Set 2

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September 14, 2011

Question 1.

To show that $\mathcal{H}om(\mathscr{F},\mathscr{G})(U) := \mathrm{Mor}(\mathscr{F}|_U,\mathscr{G}|_U)$ is a sheaf of sets on X, we first construct a restriction map and show that it is a presheaf. Given $U \subset V$, we let $res_{V,U} : \mathcal{H}om(\mathscr{F},\mathscr{G})(V) \to \mathcal{H}om(\mathscr{F},\mathscr{G})(U)$ by mapping $\phi \mapsto \psi$, where $\psi \in \mathcal{H}om(\mathscr{F},\mathscr{G})(U)$ is the data of maps that given any open set $U_0 \subset U$, we let the map $\psi(U_0)$ be the map $\phi(U_0)$. These data of maps $\psi(U)$ "behaves well" because the ϕ behave well.

Given $U \subset V \subset W$, we will check that $res_{V,U} \circ res_{W,V} = res_{W,U}$. Suppose $f \in \mathcal{H}om(\mathscr{F},\mathscr{G})(W)$, then $res_{W,V} f \in \mathcal{H}om(\mathscr{F},\mathscr{G})(V)$ is the data of maps that for all open $V_0 \subset V$, $res_{W,V} f(V_0) = f(V_0)$. Therefore, $res_{V,U} \circ res_{W,V} f$ is the data of maps that for all open $U_0 \subset U$, $res_{V,U} \circ res_{W,V} f(U_0) = res_{W,V} f(U_0) = f(U_0)$. Thus $res_{V,U} \circ res_{W,V} = res_{W,U}$.

In order to check that $\mathcal{H}om(\mathscr{F},\mathscr{G})(U) := \mathrm{Mor}(\mathscr{F}|_U,\mathscr{G}|_U)$ is a sheaf of sets on X, we will check that the construction satisfies the identiy axiom and gluability axiom. First let $U = \bigcup_i U_i$, and let $f_1, f_2 \in \mathcal{H}om(\mathscr{F},\mathscr{G})(U)$ be two elements such that $res_{U,U_i} f_1 = res_{U,U_i} f_2$ for all i. Then given any open $V \subset U$, we want to show that $f_1(V) = f_2(V)$. To achieve this, we consider the diagram (given any U_i)

$$\begin{array}{ccccccc}
\mathscr{F}(V) & = & \mathscr{F}|_U(V) & \xrightarrow{\quad f_1(V) \quad} & \mathscr{G}|_U(V) & = & \mathscr{G}(V) \\
 & & \downarrow & & \downarrow & & \\
\mathscr{F}(V \cap U_i) & = & \mathscr{F}|_{U_i}(V \cap U_i) & \xrightarrow{f_1(V \cap U_i)} & \mathscr{G}|_{U_i}(V \cap U_i) & = & \mathscr{G}(V \cap U_i)
\end{array}$$

where the vertical maps are the restriction maps from V to $V \cap U_i$. Replacing f_1 by f_2 will give us another such diagram. The diagram maps an element $p \in \mathscr{F}|_U(V)$ to

$$\begin{array}{ccc}
p & \xrightarrow{\quad f_1(V) \quad} & f_1(V)(p) \\
\downarrow & & \downarrow \\
res_{V,V\cap U_i}(p) & \xrightarrow{f_1(V \cap U_i)} & f_1(V \cap U_i)(res_{V,V\cap U_i}(p)) = res_{V,V\cap U_i}(f_1(V)(p))
\end{array}$$

Now as $res_{U,U_i} f_1 = res_{U,U_i} f_2$ for all i, we see that

$$\begin{aligned}
res_{V,V\cap U_i}(f_1(V)(p)) &= f_1(V \cap U_i)(res_{V,V\cap U_i}(p)) \\
= f_2(V \cap U_i)(res_{V,V\cap U_i}(p)) &= res_{V,V\cap U_i}(f_2(V)(p))
\end{aligned}$$

for all i. Now since $\mathscr{G}$ is a sheaf, we see that the elements $f_1(V)(p), f_2(V)(p) \in \mathscr{G}|_U(V)$ must actually be equal (by the identity axiom on $\mathscr{G}$). So $f_1(V)(p) = f_2(V)(p)$ and thus $f_1(V) = f_2(V)$. Since $V \subset U$ is arbitrary, we conclude that $f_1 = f_2 \in \mathcal{H}om(\mathscr{F},\mathscr{G})(U)$.

Next we check the gluability axiom. Again suppose $U = \bigcup_i U_i$ and given $f_i \in \mathcal{H}om(\mathscr{F},\mathscr{G})(U_i)$ for all i such that $res_{U_i,U_i\cap U_j} f_i = res_{U_j,U_i\cap U_j} f_j$, then we would like to show that there is some $f \in \mathcal{H}om(\mathscr{F},\mathscr{G})(U)$

such that $res_{U,U_i} f = f_i$ for all i. Let $V \subset U = \bigcup_i U_i$, then $V = \bigcup_i (U_i \cap V) = \bigcup_i V_i$ where $V_i = V \cap U_i$. Now consider the diagram

$$\begin{array}{ccc}
\mathscr{F}|_U(V) & & \\
{\scriptstyle res_{V,V_i}}\downarrow & & \\
\mathscr{F}|_{U_i}(V_i) & \xrightarrow{f_i(V_i)} & \mathscr{G}|_{U_i}(V_i) \\
{\scriptstyle res_{V_i,V_i\cap V_j}}\downarrow & & {\scriptstyle res_{V_i,V_i\cap V_j}}\downarrow \\
\mathscr{F}|_{U_i\cap U_j}(V_i \cap V_j) & \xrightarrow{f_i(V_i \cap V_j)} & \mathscr{G}|_{U_i\cap U_j}(V_i \cap V_j)
\end{array}$$

Replacing some of the i with j will give a diagram for the case $res_{U_j,U_i\cap U_j} f_j$ (to save room and effort it will not be shown here). Given an element $p \in \mathscr{F}|_U(V)$, following the diagram above, we have

$$\begin{array}{ccc}
p & & \\
\downarrow & & \\
res_{V,V_i} p & \xrightarrow{f_i(V_i)} & f_i(V_i) \circ res_{V,V_i}(p) \\
\downarrow & & \downarrow \\
res_{V,V_i\cap V_j} p & \xrightarrow{f_i(V_i \cap V_j)} & f_i(V_i \cap V_j) \circ res_{V,V_i\cap V_j}(p) = res_{V_i,V_i\cap V_j} \circ f_i(V_i) \circ res_{V,V_i}(p)
\end{array}$$

Now as

$$\begin{aligned}
res_{V_i,V_i\cap V_j} \circ f_i(V_i) \circ res_{V,V_i}(p) &= f_i(V_i \cap V_j) \circ res_{V,V_i\cap V_j}(p) \\
= f_j(V_i \cap V_j) \circ res_{V,V_i\cap V_j}(p) &= res_{V_j,V_i\cap V_j} \circ f_j(V_j) \circ res_{V,V_j}(p)
\end{aligned}$$

we see by the gluability of $\mathscr{G}$ that there exist a $g \in \mathscr{G}|_U(V)$ such that $res_{V,V_i} g = f_i(V_i) \circ res_{V,V_i}(p)$ for all i. We define a map $f(V) : \mathscr{F}|_U(V) \to \mathscr{G}|_U(V)$ by taking $p \mapsto g$. Now since $V \subset U$ is arbitrary, it follows from our construction that we have a $f \in \mathcal{H}om(\mathscr{F}, \mathscr{G})(U)$ such that $res_{U,U_i} f = f_i$ for all i. Thus we conclude that the "sheaf hom" is a sheaf.

Finally, if $\mathscr{G}$ is a sheaf of abelian group, then given any $f, g \in \mathcal{H}om(\mathscr{F}, \mathscr{G})(U)$ and any $V \subset U$, we can define the sum $f + g$ to be the data of maps $(f + g)(V) = f(V) + g(V)$. Since $\mathscr{G}$ is a sheaf of abelian group, we know that $\mathrm{Hom}(\mathscr{F}, \mathscr{G})$ is also an abelian group, and so the addition operation as defined makes $\mathcal{H}om(\mathscr{F}, \mathscr{G})(U)$ into a sheaf of abelian groups.

Question 2.

We will show that $ker_{pre}\phi$ is a presheaf. Consider the following diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & ker_{pre}\phi(V) & \xrightarrow{i} & \mathscr{F}(V) & \xrightarrow{\phi(V)} & \mathscr{G}(V) \\
 & & \vdots\,\exists! & & \downarrow{\scriptstyle res_{V,U}} & & \downarrow{\scriptstyle res_{V,U}} \\
0 & \longrightarrow & ker_{pre}\phi(U) & \xrightarrow{i} & \mathscr{F}(U) & \xrightarrow{\phi(U)} & \mathscr{G}(U)
\end{array}$$

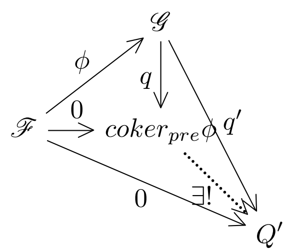
The horizontal maps form exact sequences. We would like to show that there exist a unique map $ker_{pre}\phi(V) \to ker_{pre}\phi(U)$, which we will take to be the restriction map. Consider an element $p \in ker_{pre}\phi(V)$. We see that

$$\begin{array}{ccccc}
p & \xrightarrow{i} & i(p) & \xrightarrow{\phi(V)} & \phi(V) \circ i(p) = \quad 0 \\
 & & \downarrow & & \downarrow \\
 & & res_{V,U}(i(p)) & \xrightarrow{\phi(U)} & res_{V,U}(0) = \quad 0
\end{array}$$

Thus we see that $\phi(U)(res_{V,U}(i(p))) = 0$, and hence by exacness, $res_{V,U}(i(p)) = i(p')$ for some unique p'. Thus the map $ker_{pre}\phi(V) \to ker_{pre}\phi(U)$ which maps $p \mapsto p'$ is the unique map such that the first diagram commutes. We take this to be the restriction map of $ker_{pre}\phi$. Now $res_{U,U}$ is the identity map because the other two restriction maps in the first diagram are both the identity map. Thus it remains to check that if $U \subset V \subset W$, then $res_{V,U} \circ res_{W,V} = res_{W,U}$. Consider any $p \in ker_{pre}\phi(W)$, then $res_{V,U} \circ res_{W,V}(p) = res_{V,U}p'$ where p' is the unique element such that $res_{W,V}(i(p)) = i(p')$. However, by our construction, $res_{V,U}p' = p''$ where p'' is the unique element such that $res_{V,U}(i(p')) = i(p'')$. Now $i(p'') = res_{V,U}(i(p')) = res_{V,U} \circ res_{W,V}(i(p)) = res_{W,U}(i(p))$ by the presheaf property of $\mathscr{F}$. Thus $res_{V,U} \circ res_{W,V}(p) = res_{W,U}(p)$ for all p, and thus $res_{V,U} \circ res_{W,V} = res_{W,U}$, and we conclude that $ker_{pre}\phi$ is a presheaf.

Question 3.

We will show that the presheaf cokernel satisfies the universal property of cokernels in the category of sheaves. Suppose we are given a presheaf morphism $\phi : \mathscr{F} \to \mathscr{G}$. What we want to show is that given



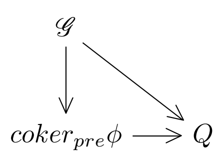
there exist a unique presheaf morphism $\Psi : coker_{pre}\phi \to Q'$. Given any open U, we see that there is a unique map $\psi_U : coker_{pre}\phi(U) \to Q'(U)$ by the universal property of cokernel in the abelian category. So we take Ψ to be the data of maps such that for any open U, $\Psi(U) = \psi_U$. To verify this is actually a presheaf morphism, we need to check that given $U \subset V$, the square

$$\begin{array}{ccc} coker_{pre}\phi(V) & \xrightarrow{\Psi(V)} & Q(V) \\ {\scriptstyle res}\downarrow & & \downarrow{\scriptstyle res} \\ coker_{pre}\phi(U) & \xrightarrow{\Psi(U)} & Q(U) \end{array}$$

commutes. Given any $p \in coker_{pre}\phi(V)$, we see, by the exactness of the sequence

$$\mathscr{F}(V) \xrightarrow{\phi(V)} \mathscr{G}(V) \xrightarrow{q(V)} coker_{pre}\phi(V) \to 0$$

that there is an element $x \in \mathscr{G}(V)$ such that $q(V)(x) = p$. By the commutativity of the triangle



we see that $q'(V)(x) = \Psi(V)(p)$. Now as the following two squares are commutative

$$\begin{array}{ccc} \mathscr{G}(V) & \to & Q(V) \\ \downarrow & & \downarrow \\ \mathscr{G}(U) & \to & Q(U) \end{array} \qquad \begin{array}{ccc} \mathscr{G}(V) & \to & coker_{pre}\phi(V) \\ \downarrow & & \downarrow \\ \mathscr{G}(U) & \to & coker_{pre}\phi(U) \end{array}$$

we see that

$$res_{V,U} \circ \Psi(V)(p) = res_{V,U} \circ q'(V)(x) = q'(U) \circ res_{V,U}(x) = \Psi(U) \circ q(U) \circ res_{V,U}(x) = \Psi(U) \circ res_{V,U}(p)$$

and thus the square

$$\begin{array}{ccc} coker_{pre}\phi(V) & \xrightarrow{\Psi(V)} & Q(V) \\ {\scriptstyle res}\downarrow & & {\scriptstyle res}\downarrow \\ coker_{pre}\phi(U) & \xrightarrow{\Psi(U)} & Q(U) \end{array}$$

commute, and hence we see that Ψ is a presheaf morphism, and therefore $coker_{pre}\phi$ satisfies the universal property of cokernels in the category of presheaves.

Question 4.

In Question 2 we showed that $ker_{pre}\phi$ is a presheaf, so in order to show that $ker_{pre}\phi$ is in fact a sheaf, we need to verify the identity axiom and the gluability axiom. Now suppose $U = \cup U_i$, and $f_1, f_2 \in ker_{pre}\phi(U)$ and $res_{U,U_i} f_1 = res_{U,U_i} f_2$ for all i. Then from the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & ker_{pre}\phi(U) & \xrightarrow{i} & \mathscr{F}(U) & \xrightarrow{\phi(U)} & \mathscr{G}(U) \\ & & \vdots\ \exists! & & \downarrow{\scriptstyle res_{U,U_i}} & & \downarrow{\scriptstyle res_{U,U_i}} \\ 0 & \longrightarrow & ker_{pre}\phi(U_i) & \xrightarrow{i} & \mathscr{F}(U_i) & \xrightarrow{\phi(U_i)} & \mathscr{G}(U_i) \end{array}$$

we see that if $res_{U,U_i} f_1 = res_{U,U_i} f_2$ for all i, then $i \circ res_{U,U_i}(f_1) = i \circ res_{U,U_i}(f_2)$ for all i, and hence $res_{U,U_i} \circ i(f_1) = res_{U,U_i} \circ i(f_2)$ for all i by the commutativity of the left square. Now as $\mathscr{F}$ is a sheaf, we see that $i(f_1) = i(f_2)$, and thus as i is injective by exactness, $f_1 = f_2$.

Next we verify the gluability axiom. Suppose $U = U_i$, and given $f_i \in ker_{pre}\phi(U_i)$ for all i such that $res_{U_i,U_i\cap U_j} f_i = res_{U_j,U_i\cap U_j} f_j$ for all i, j, then we have $i(res_{U_i,U_i\cap U_j} f_i) = i(res_{U_j,U_i\cap U_j} f_j)$ for all i, j (sorry for the notation, the map i is again the map as in the diagram above), so by the commutativity of the left square we have $res_{U_i,U_i\cap U_j} \circ i(f_i) = res_{U_j,U_i\cap U_j} \circ i(f_j)$ for all i, j. By the gluability of $\mathscr{F}$, we see that there is an element $f' \in \mathscr{F}(U)$ such that $res_{U,U_i} f' = i(f_i)$ for all i. We will show that $\phi(U)$ maps $f' \mapsto 0$. Consider first the element $\phi(U)(f')$, we see that $res_{U,U_i} \circ \phi(U)(f') = \phi(U_i) \circ res_{U,U_i}(f') = \phi(U_i) \circ i(f_i) = 0$ for all i. Thus we see that $\phi(U)(f')$ restricts to 0 for all U_i. Now as $\mathscr{G}$ is a sheaf, the identity axiom and the gluability axiom shows that 0 is the only element in $\mathscr{G}(U)$ such that it restricts to 0 for all U_i, and thus $\phi(U)(f') = 0$, and hence f' is in the kernel of $\phi(U)$, so by exactness there is a unique element $f \in ker_{pre}\phi(U)$ such that $i(f) = f'$, and we see that this f is the desired element in $ker_{pre}\phi(U)$ such that $res_{U,U_i} f = f_i$ for all i. This concludes the proof that $ker_{pre}\phi$ is a sheaf.

Since $ker_{pre}\phi$ satisfies the universal property in the category of presheaves (follows from a similar proof as Question 3.), and since the category of sheaves on X is a full subcategory of presheaves on X, we can immediately conclude that the sheaf $ker_{pre}\phi$ satisfies the universal property of kernels in the category of sheaves.

Question 5.

We wil first verify that the sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{j} \mathscr{O}_X \xrightarrow{exp} \mathscr{F} \longrightarrow 0$$

given by letting j be the natural inclusion and letting $exp : f \mapsto exp 2\pi i f$ is exact. We will do so by showing that the following is exact

$$0 \longrightarrow \underline{\mathbb{Z}}(U) \xrightarrow{j(U)} \mathscr{O}_X(U) \xrightarrow{exp(U)} \mathscr{F}(U) \longrightarrow 0$$

for all U. The exactness at $\underline{\mathbb{Z}}(U)$ is clear because locally constant functions (with constants in the integers) are holomorphic, so if two locally constant functions (with constants in the integers), when treated as holomorphic functions, are the same, then they must be the same when treated as locally constant functions (with constants in the integers). Next we verify the exactness at $\mathscr{O}_X(U)$. First, given $f \in \underline{\mathbb{Z}}(U)$, we see that $exp \circ j(f) = exp 2\pi i j(f) = 1$ as $j(f)$ has values in $\mathbb{Z}$. This shows that $Im(j(U)) \subset ker(exp(U))$. On the other hand, suppose a holomorphic function $f \in \mathscr{O}_X(U)$ is such that $exp 2\pi i f = 1$, then for any $x \in U$, we see that $2\pi i f'(x) = 1'(x)/1(x) = 0$ and so the derivative vanishes for all x, which implies that f is locally constant. Together with the fact that $exp 2\pi i f = 1$, we see that f is a locally constant functions (with constants in the integers), so $Im(j(U)) \supset ker(exp(U))$ and thus $Im(j(U)) = ker(exp(U))$. Finally, given a function g admitting a holomorphic logarithm, by the very definition of having a holomorphic logarithm, it means that there is a holomorphic function f such that $g = exp(f)$ for all $z \in U$. Thus it is in the image of $exp(U)$, and so the sequence is also exact at $\mathscr{F}(U)$. This concludes the proof that the sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{j} \mathscr{O}_X \xrightarrow{exp} \mathscr{F} \longrightarrow 0$$

is exact.

$\mathscr{F}$ is not a sheaf because it fails the gluability axiom. From complex analysis, we see that there is no function that has a holomorphic logarithm defined globally on $\mathbb{C}$, and thus $\mathscr{F}(\mathbb{C}) = \emptyset$. However, there are functions such that locally, logarithms exists (like the constant non-zero function for). Thus we can not "glue" functions that locally admit holomorphic logarithms. Hence $\mathscr{F}$ is not a sheaf.