Math 256A Problem Set 10

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Question 1.

For $n \geq 0$, a global section in $\Gamma(\mathbb{P}^1, \mathscr{O}(n))$ is a polynomial $f(x_{1/0}) \in k[x_{1/0}]$ and a polynomial $g(x_{0/1}) \in k[x_{0/1}]$ such that $f(1/x_{0/1})x_{0/1}^n = g(x_{0/1})$. This implies that $f(x_{1/0}) = a_n x_{1/0}^n + a_{n-1}x_{1/0}^{n-1} + ... + a_1 x_{1/0} + a_0$ and $g(x_{0/1}) = a_n + a_{n-1}x_{0/1}^1 + ... + a_1 x_{0/1}^{n-1} + a_0 x_{0/1}^n$. Thus we see that

$$\dim \Gamma(\mathbb{P}^1, \mathscr{O}(n)) = (n+1) \text{ for } n \geq 0$$

On the other hand, for $n < 0$, we see that the transition function from $k[x_{1/0}] \to k[x_{0/1}]$ is multiplying by $x_{0/1}^n$ because for $n < 0$ it is the dual bundle of $\mathscr{O}(1)^{\otimes -n}$, and the transition function of $(\mathscr{O}(1)^{\otimes -n})^\vee$ is the inverse of the transition function of $\mathscr{O}(1)^{\otimes -n}$ by 14.1.C in Vakil. Now a global section in $\Gamma(\mathbb{P}^1, \mathscr{O}(n))$ is a polynomial $f(x_{1/0}) \in k[x_{1/0}]$ and a polynomial $g(x_{0/1}) \in k[x_{0/1}]$ such that $f(1/x_{0/1})x_{0/1}^n = g(x_{0/1})$. But because $n < 0$ is negative, we see that no such f and g can ever exist because the degree of $x_{0/1}$ in f will always be negative. Thus since no f and g can satisfy the condition, we conclute that

$$\dim \Gamma(\mathbb{P}^1, \mathscr{O}(n)) = 0 \text{ for } n < 0$$

Question 2.

If m or n (or both) is positive and $m \neq n$, then the previous exercise show that the dimension of the global sections are not equal, and hence $\mathscr{O}(n)$ is not isomorphic to $\mathscr{O}(m)$. The remaining case is when m and n are both negative and $m \neq n$. Suppose $\mathscr{O}(n) \cong \mathscr{O}(m)$, i.e. $\mathscr{O}(-n)^\vee \cong \mathscr{O}(-m)^\vee$, then $\mathscr{O}(-n) \cong \mathscr{O}(-n)^{\vee\vee} \cong \mathscr{O}(-m)^{\vee\vee} \cong \mathscr{O}(-m)$ (by 14.1.C in Vakil again) which contradicts the first part of this question. Thus we conclude that if $m \neq n$, then $\mathscr{O}(n)$ is not isomorphic to $\mathscr{O}(m)$.

Question 3.

Similar to the case of $\mathbb{P}^1$, we see that a global section in $\Gamma(\mathbb{P}^m, \mathscr{O}_{\mathbb{P}^m}(n))$ is a homogeneous polynomial of degree n in $m+1$ variables. Thus we only need to find how many different ways can $x_1^{i_1} x_2^{i_2} ... x_{m+1}^{i_{m+1}}$ be of degree n. This is the same as the combinatorics problem of putting n balls in $m+1$ boxes, and the combinatorial solution to this is $\begin{pmatrix} n+(m+1)-1 \\ n \end{pmatrix} = \begin{pmatrix} n+m \\ n \end{pmatrix}$. Thus we see that $\dim \Gamma(\mathbb{P}^m, \mathscr{O}_{\mathbb{P}^m}(n)) = \begin{pmatrix} n+m \\ n \end{pmatrix}$.