Math 256A Problem Set 1

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September 7, 2011

Question 1.

We will show that the functor
$$h : \mathcal{C} \to Fun(\mathcal{C}^{op}, Set)$$
gives a bijection
$$h_{X,Y} : Hom_{\mathcal{C}}(X,Y) \to Hom_{Fun(\mathcal{C}^{op},Set)}(h(X), h(Y))$$
for all $X, Y \in \mathcal{C}$.

Given any arbitrary $X, Y \in \mathcal{C}$ and $f \in Hom_{\mathcal{C}}(X,Y)$, we let m be a natural transformation from h_X to h_Y defined as follows: Given any $A \in \mathcal{C}$, we let $m_A : h_X(A) \to h_Y(A)$ by taking $h \mapsto f \circ h$ where $h \in Hom(A,X) = h_X(A)$. Suppose $g : A \to A'$, then we see that the diagram

$$\begin{tikzcd} h_X(A) \arrow[d, "m_A"'] & h_X(A') \arrow[l, "h_X(g)"'] \arrow[d, "m_{A'}"'] \\ h_Y(A) & h_Y(A') \arrow[l, "h_Y(g)"'] \end{tikzcd} \quad \text{is given by} \quad \begin{tikzcd} h \circ g \arrow[d, "m_A"'] & h \arrow[l, "h_X(g)"'] \arrow[d, "m_{A'}"'] \\ f \circ h \circ g & f \circ h \arrow[l, "h_Y(g)"'] \end{tikzcd}$$

and thus the diagram commutes.

On the other hand, given any natural transformation $m \in Hom_{Fun(\mathcal{C}^{op},Set)}(h(X), h(Y))$, we consider the map $m_X : Hom_{\mathcal{C}}(X,X) \to Hom_{\mathcal{C}}(X,Y)$, and let $f = m_X(Id)$ where $Id \in Hom_{\mathcal{C}}(X,X)$ is the identity map. Then $m_X(Id) \in Hom_{\mathcal{C}}(X,Y)$.

The two paragraphs above show that we can obtain a $\mathcal{C}$-morphism $f \in Hom_{\mathcal{C}}(X,Y)$ from a natural transformation $m \in Hom_{Fun(\mathcal{C}^{op},Set)}(h(X), h(Y))$ and vice versa. It remains to show that these two constructions are "inverses" of each other. Given any $m \in Hom_{Fun(\mathcal{C}^{op},Set)}(h(X), h(Y))$, the $\mathcal{C}$-morphism we get is $m_X(Id) : X \to Y$. From this $m_x(Id)$ we get a natural transformation $n \in Hom_{Fun(\mathcal{C}^{op},Set)}(h(X), h(Y))$ sending (for any given $A \in \mathcal{C}$ and $f_1 \in h_X(A)$) $f_1 \mapsto m_X(Id) \circ f_1 = m_A(f_1)$ where the last equality is true because m is natural so the diagram commutes:

$$\begin{tikzcd} h_X(A) \arrow[d, "m_A"'] & h_X(X) \arrow[l, "h_X(f_1)"'] \arrow[d, "m_X"'] \\ h_Y(A) & h_Y(X) \arrow[l, "h_Y(f_1)"'] \end{tikzcd} \quad \text{is given by} \quad \begin{tikzcd} Id \circ f_1 \arrow[d, "m_A"'] & Id \arrow[l, "h_X(f_1)"'] \arrow[d, "m_X"'] \\ m_X(Id) \circ f_1 & m_X(Id) \arrow[l, "h_Y(f_1)"'] \end{tikzcd}$$

so $n = m$ and we get back the original natural transformation m.

On the other hand, given any $f \in Hom_{\mathcal{C}}(X,Y)$, the natural transformation we get is $f \circ ()$, the natural transformation given by left composition by f. From this natural transformation, we get back the $\mathcal{C}$-morphism $f \circ (Id) = f$.

What this shows is that there is a bijection
$$h_{X,Y} : Hom_{\mathcal{C}}(X,Y) \to Hom_{Fun(\mathcal{C}^{op},Set)}(h(X), h(Y))$$

for all $X, Y \in \mathcal{C}$, so the association $A \mapsto h_A$ defines a fully faithful functor

$$h : \mathcal{C} \to Fun(\mathcal{C}^{op}, Set)$$

Question 2.

Let the k-algebra S be $k[\underline{X}]/(f_1(\underline{X}), f_2(\underline{X}), ..., f_r(\underline{X}))$. We will show that $h_S \cong F$ by showing that given a k-algebra $g : k \to R$, there is a bijection (isomorphism of sets)

$$Hom(S, R) \leftrightarrow \{\underline{a} \in R^m | f_i^g(\underline{a}) = 0 \text{ for all } i\}$$

Let $\pi : k[\underline{X}] \to S$ be the projection map, then π is surjective with kernel $(f_1(\underline{X}), f_2(\underline{X}), ..., f_r(\underline{X}))$. Thus the map $\pi^* : Hom(S, R) \to Hom(k[\underline{X}], R)$ is injective, and the image of π^* are exactly those morphisms $\xi : k[\underline{X}] \to R$ such that $(f_1(\underline{X}), f_2(\underline{X}), ..., f_r(\underline{X})) \subset ker(\xi)$. However, any k-algebra homomorphism $\xi : k[\underline{X}] \to R$ can be identified with $(\xi(X_1), \xi(X_2), ..., \xi(X_m)) \in R^m$ because ξ is uniquely determined by where it sends $X_1, ..., X_n$, and also any ξ determines a number $(\xi(X_1), \xi(X_2), ..., \xi(X_m)) \in R^m$. So

$$Hom(k[\underline{X}], R) \leftrightarrow \{\underline{a} \in R^m\}$$

is a bijection.

Since the ideal $(f_1(\underline{X}), f_2(\underline{X}), ..., f_r(\underline{X})) = \{r_1 f_1 + r_2 f_2 + ... + r_m f_m | r_m \in k[\underline{X}]\}$, we can infer from this that $(f_1(\underline{X}), f_2(\underline{X}), ..., f_r(\underline{X})) \subset ker(\xi)$ if and only if $f_i^g((\xi(X_1), \xi(X_2), ..., \xi(X_m))) = 0$ for all i. Therefore the sets below are bijections

$$Hom(S, R) \leftrightarrow \{\xi : k[\underline{X}] \to R : (f_1(\underline{X}), f_2(\underline{X}), ..., f_r(\underline{X})) \subset ker(\xi)\} \leftrightarrow \{\underline{a} \in R^m | f_i^g(\underline{a}) = 0 \text{ for all } i\}$$

This shows that for this S, we have an isomorphism of functors $i : h_S^{op} \cong F$. To show that the (S, i) are unique up to unique isomorphism, we use Question 1, which is Yoneda's lemma. Yoneda's Lemma state that

$$h : \mathcal{C} \to Fun(\mathcal{C}^{op}, Set)$$

is a fully faithful functor. Thus if some other pair (T, j) also satisfies $j : h_T^{op} \cong F$, we have $h_S \cong h_T$, and using Yoneda's lemma, $S \cong T$ and the isomorphism is unique.

Question 3.

1. Given that F is representable, and (S, i) is the pair with $S \in \mathcal{C}$ and $i : h_S \cong F$, then let $Id \in h_S(S) = Hom(S, S)$ be the identity morphism, we see that $i_S(Id) \in F(S)$, so $F(S)$ is not empty. Since $F(S)$ is not an empty set, we see that there exist a commutative diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d] & Y \arrow[d, "g"'] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

2. Let $S = \{(x, y) \in X \times Y | f(x) = g(y)\}$. We will show that given $f : X \to Z$ and $g : Y \to Z$, F can be represented by this S. Given any $W \in Set$, and any $h \in Hom_{Set}(W, S)$, we define a natural transformation $i : h_S \to F$ at W by $i_W : h \mapsto (p_x \circ h, p_y \circ h)$, where p_x, p_y are projection maps from S to its first and second factor respectively. This is a natural transformation because given $g : W_1 \to W_2$, the following maps commute:

$$\begin{tikzcd} h_X(W_1) \arrow[d, "i_{W_1}"'] & h_X(W_2) \arrow[l, "h_X(g)"'] \arrow[d, "i_{W_2}"'] \\ F(W_1) & F(W_2) \arrow[l, "F(g)"'] \end{tikzcd} \quad \text{Given by} \quad \begin{tikzcd} h \circ g \arrow[d, "i_{W_1}"'] & h \arrow[l] \arrow[d, "i_{W_2}"'] \\ (p_x \circ h \circ g, p_y \circ h \circ g) & (p_x \circ h, p_y \circ h) \arrow[l] \end{tikzcd}$$

It remains to show that for each W, the map i_W is an isomorphism (i.e. a bijection in Set). We will construct an inverse j_W to i_W, as follows: Let $(a,b) \in F(W)$, then we let $h : W \to S$ by mapping r to $(a(r), b(r))$. This function is well defined by construction, and $(a(r), b(r))$ is in S. We need to show that $j_W \circ i_W$ and $i_W \circ j_W$ are the identity functions respectively. The first one takes a function h to a pair of maps $(p_x \circ h, p_y \circ h)$, which is taken to the function $g : w \mapsto (p_x \circ h(w), p_y \circ h(w)) = h(w)$, so $g = h$. On the other hand, the second one takes a pair of functions $(a,b) \in F(W)$ to a map $h \in Hom(W,S)$ by taking $r \mapsto (a(r), b(r))$, and this map is taken back to a pair $(p_x \circ h, p_y \circ h) = (a,b)$ by our construction. Thus given f, g, F is represented by this S. Since f, g are arbitrary, Set has fiber products.

As for the case Set^{op}, we consider $S = (X \coprod Y)/\{f(z) \sim g(z)\}$, the disjoint union of X and Y with the points $f(x), g(x)$ identified. Given $h \in Hom(S,B)$, we get a pair of maps $(h \circ i_x, h \circ i_y)$ where $i_x : X \to X \coprod Y$ and $i_y : Y \to X \coprod Y$ are the obvious maps. Then we see that $h \circ i_x(f(z)) = h(g(z)) = h \circ i_y(g(z))$, so $h \circ i_x \circ f = h \circ i_y \circ g$. On the other hand, given $a : X \to B$ and $b : Y \to B$, we define $h \in Hom(S,B)$ by mapping $s \mapsto a(x)$ if $s = i_x(x)$ and mapping $s \mapsto b(y)$ if $s \in i_y(y)$. This is a well defined function, as points $f(z), g(z)$ are identified. Now given $h \in Hom(S,B)$, we get a pair of maps $(h \circ i_x, h \circ i_y)$, and from our construction we get back the original map $h : S \to B$. Given $a : X \to B$ and $b : Y \to B$, we get a map $h \in Hom(S,B)$ by mapping $s \mapsto a(x)$ if $s = i_x(x)$ and mapping $s \mapsto b(y)$ if $s \in i_y(y)$, from which we recover the pair of maps (a,b). Thus S is the representing object for the given f, g in the category Set^{op}. Since f, g are arbitrary, Set^{op} has fiber products.

3. Consider first the forgetful functor from Ring to Set, then we see that if F is representable for any given f and g, then as sets, the representation S must be of the form $S = \{(x,y) \in X \times Y | f(x) = g(y)\}$. Hence we define $S = \{(x,y) \in X \times Y | f(x) = g(y)\}$, where f, g are commutative ring homomorphisms from commutative rings X to Y. Now we check that this S is a commutative ring: Given any (x_1, y_1) and $(x_2, y_2) \in S$, we see that $f(x_1 - x_2) = f(x_1) - f(x_2) = g(y_1) - g(y_2) = g(y_1 - y_2)$ and $f(x_1 x_2) = f(x_1) f(x_2) = g(y_1) g(y_2) = g(y_1 y_2)$, so S is a subring of $X \times Y$, and thus $S \in Ring$. Now using a very similar argument as the previous part of this problem we see that S is the representing object, and thus Ring has fiber products.

4. Claim that the commutative ring $S = X \otimes_Z Y$ satisfies $h_S \cong F$ in $Ring^{op}$. First of all, since X, Y are both commutative rings and $f : Z \to X$ and $g : Z \to Y$, we see that X, Y are both Z-algebras. We can then form a tensor product $X \otimes_Z Y$, and this product has a well-defined multiplication $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$, which makes $X \otimes_Z Y$ into a commutative ring (and also a Z-algebra).

   We will show that there is a bijection of sets between $\{(a : X \to W, b : Y \to W) | a \circ f = b \circ g\}$ and $Hom(S, W)$. Given any $(a,b) \in F(W) = \{(a : X \to W, b : Y \to W) | a \circ f = b \circ g\}$, we have the following Z-bilinear map

$$X \times Y \xrightarrow{(a,b)} W \times W \xrightarrow{\text{multiplication in W}} W$$

   which gives a Z-module map $X \otimes_Z Y \xrightarrow{v} C$ by taking $v(x \otimes y) = a(x)b(y)$. Now as elements in $a(X)$ and $b(Y)$ commute (being elements of a commutative ring W), we see that the map $v : X \otimes_Z Y \longrightarrow C$ is actually a ring homomorphism, since

$$\begin{aligned} v((x_1 \otimes y_1)(x_2 \otimes y_2)) &= v(x_1 x_2 \otimes y_1 y_2) \\ &= a(x_1 x_2) b(y_1 y_2) \\ &= a(x_1) a(x_2) b(y_1) b(y_2) \\ &= a(x_1) b(y_1) a(x_2) b(y_2) \\ &= v(x_1 \otimes y_1) v(x_2 \otimes y_2) \end{aligned}$$

   Thus given (a,b) we have constructed a map $v \in Hom(S, W)$.

   On the other hand, given a commutative ring homomorphism $v : S \to W$, we define $a : X \to W$ by $a : x \mapsto v(x \otimes 1)$ and $b : Y \to W$ by $b : y \mapsto v(1 \otimes y)$. The maps a, b satisfies $a \circ f = b \circ g$ by construction, as $a(f(z)) = v((z \cdot 1) \otimes 1) = v(1 \otimes (z \cdot 1)) = b(g(z))$.

The maps above gives a bijection: Given $v \in Hom(S,W)$, our construction sends v to the map taking (x,y) to $(v(x\otimes 1), v(1\otimes y)) \mapsto v(x\otimes y)$, and this map is in turn send to the map h taking $h(x\otimes y) = v(x\otimes 1)v(1\otimes y) = v(x\otimes y)$, thus the composition of the two constructions gives back v. On the other hand, if we start out with (a,b), then we get a map $v(x\otimes y) = a(x)b(y)$, and this map gives us a pair (a',b') by $a' : x \mapsto v(x\otimes 1) = a(x)$ and $b' : y \mapsto v(1\otimes y) = b(y)$. Thus $(a',b') = (a,b)$ and we recover the pair (a,b) again. Thus there is a bijection between $\{(a : X \to W, b : Y \to W) | a\circ f = b\circ g\}$ and $Hom(S,W)$. Thus S is the representing object for the given f,g in the category $Ring^{op}$. Since f,g are arbitrary, $Ring^{op}$ has fiber products.

Question 4.

Consider the functor $h_{(B/I)\otimes_A C}$ and $h_{(B\otimes_A C)/I^e}$. We will find out what these two functors represent. The functor $h_{X\otimes_A Y}$ for any given $X,Y \in Ring$ is the pushout in the category $Ring$ in Question 3-4. On the other hand, X/Y for any $X \in Ring$ and $Y \subset X$ an ideal is a representation of the functor $F : Ring \to Set$ by taking $W \mapsto \{p : X \to W | Y \subset ker(p)\}$. Thus we see that $h_{(B/I)\otimes_A C}$ represents the functor F sending $W \mapsto \{(a : B/I \to W, b : C \to W) | a\circ p\circ f = b\circ g\}$, (where $p : B \to B/I$ is the projection map), which further represents the functor F' sending $W \mapsto \{(a' : B \to W, b : C \to W) | a'\circ f = b\circ g \text{ and } I \subset ker(a')\}$. On the other hand, we see that $h_{(B\otimes_A C)/I^e}$ represents the functor G sending $W \mapsto \{d : B\otimes_A C \to | I^e \subset ker(d)\}$, which further represents the functor G' sending $W \mapsto \{(a : B \to W, b : C \to W) | a\circ f = b\circ g \text{ and } I \subset ker(a)\}$. Thus it is clear that $G' \cong F'$, which implies that $h_{(B/I)\otimes_A C} \cong h_{(B\otimes_A C)/I^e}$, and thus by Yoneda's lemma, which is proved in Question 1, we conclude that

$$(B/I)\otimes_A C \cong (B\otimes_A C)/I^e$$

Question 5.

Let

$$A = \left\{ \underline{a} = (a_1, a_2, ...) \in \prod_{i=1}^{\infty} G_i : a_i = \pi_{i+1}(a_{i+1}) \forall i \in \mathbb{N} \right\}$$

Claim that this is the representing object for $\varprojlim G_n$. We will show this by showing that given any $B \in Gp$, there is a bijection between $Hom_{Gp}(B,A)$ and $\varprojlim G_n(B)$.

We note first that if $p_n : A \to G_n$ are the projection of A to the n-th factor, then by our construction of A, we have $p_n = \pi_{n+1}\circ p_{n+1}$ for all n.

Given any $f \in Hom_{Gp}(B,A)$, we let $h_n = p_n \circ f : B \to G_n$ for all $n \in \mathbb{N}$, where the maps p_n are the projection $A \to G_n$ to the n-th factor. For $n \geq 2$, we see that $h_{n-1} = p_{n-1}\circ f = \pi_n \circ p_n \circ f = \pi_n \circ h_n$. Thus we obtain an element of $\varprojlim G_n(B)$.

On the other hand, given a set of collection of maps $\{h_n : B \to G_n\}_{n=1}^{\infty}$ such that $h_{n-1} = \pi_n \circ h_n$ for all $n \geq 2$, we obtain a map $f : B \to A$ by taking $f(b) = (h_1(b), h_2(b), ..., h_n(b), ...)$. The image is in A because $h(b)_i = h_i(b) = \pi_{i+1}h_{i+1}(b) = \pi_{i+1}h(b)_{i+1}$ for all $i \in \mathbb{N}$. This is a group homomorphism since each h_n is a group homomorphism for all n. Thus we obtain a homomorphism $B \to A$ for any element in $\varprojlim G_n(B)$.

The "composite" of these two constructions are "inverse" of each other, as can be seen as follows: A homomorphism $f : B \to A$ is taken to the element $\{p_n \circ f : B \to G_n\} \in \varprojlim G_n(B)$, which is taken to the function $h : B \to A$ by taking $h : b \mapsto (p_1\circ f(b), p_2\circ f(b), ..., p_n\circ f(b), ...) = f(b)$, so $h = f$ and we get back the original homomorphism f. On the other hand, an element $\{h_n : B \to G_n\} \in \varprojlim G_n(B)$ is taken to the homomorphism $f : B \to A$ by taking $f(b) = (h_1(b), h_2(b), ..., h_n(b), ...)$, from which we get a collection of maps $\{p_n \circ f\} = \{h_n\}$, which is the original collection. Thus there is a bijection between $Hom_{Gp}(B,A)$ and $\varprojlim G_n(B)$. This shows that $\varprojlim G_n$ is representable, and S is the representing object.