Math 214 Final

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Question 1.

Given a smooth manifold M, a vector field $Y : M \to TM$ on M, and a coordinate chart (U, x), the vector field Y is smooth on U if and only if the component functions with respect to this chart is smooth (Spivak chapter 5), i.e. for any $p \in U$ if we express

$$Y = \sum_i Y^i(p) \frac{\partial}{\partial x^i}\bigg|_p$$

then Y is smooth on U if and only if the Y^i are smooth.

Consider $M = \mathbb{R}^n$, we see that there is a globally defined smooth coordinate chart $(\mathbb{R}^n, (x^i))$, the standard smooth structure. Define the vector fields V_i to be $\frac{\partial}{\partial x^i}$, then $\{V_i\}_{i=1,\ldots,n}$ is linearly independent (because $\{\frac{\partial}{\partial x^i}\}_{i=1,\ldots,n}$ is linearly independent), and the remark at the beginning of the answer shows that $\{V_i\}$ generates the space of C^∞ vector fields over $\mathbb{R}^n$ as a $C^\infty(M)$-module. Thus the space of C^∞ vector fields over $\mathbb{R}^n$ is a free $C^\infty(M)$-module.

On the other hand, suppose $M = S^2$. Suppose on the contrary that C^∞ vector fields on S^2 has a linearly independent generating set as a $C^\infty(S^2)$ module. Since S^2 is locally diffeomorphic to $\mathbb{R}^2$, we see that the linearly independent generating set can only have exactly two elements. Now let V_1, V_2 be two linearly independent C^∞ vector fields on S^2. Since the hairy ball theorem tells us that $V_1|_p$ (or V_2) must be 0 for some $p \in S^2$, we see that V_1 and V_2 cannot possibly generate the whole C^∞ vector fields on S^2. Thus we have a contradiction, and we conclude that the C^∞ vector fields on S^2 is not a free $C^\infty(S^2)$-module.

Question 2.

Let M be a smooth orientable compact connected n-dimensional manifold and ω an $(n-1)$-form on M. First of all, even though it is not in the problem's original assumption on M, I think that M cannot have any boundary, because suppose $M = [1,2] \subset \mathbb{R}$ an embedded manifold with boundary and $\omega = x \in \Omega^0(M) = C^\infty(M)$, then $d\omega$ is never 0 for any $p \in [1,2]$. Thus the problem is not true if M has boundary.

Thus let M be a smooth orientable compact connected n-dimensional manifold without boundary and ω an $(n-1)$-form on M. Now by Stokes Theorem, we see that $\int_M d\omega = \int_{\partial M} \omega = 0$ since $\partial M = \varnothing$. Now since $d\omega$ is smooth and $\int_M d\omega = 0$, we see that if $d\omega(p)$ is not the zero map for all $p \in M$, in coordinates this means that $d\omega(p) = f(p) dx^1 \wedge \ldots \wedge dx^n$ is not zero the zero map for all p (which means $f(p)$ is never zero), and so by smoothness of $d\omega$ (which is equivalent to smoothness of f) it must follow that $f(p) > 0$ or $f(p) < 0$ for all $p \in M$. However, this implies that $\int_M d\omega > 0$ or $\int_M d\omega < 0$, which is a contradiction. Thus we comclude that $d\omega(p)$ must be the zero map for some $p \in M$.

Question 7-8.

(a) Define $\phi_1 = (\beta_1\psi_1 + \beta_3\psi_3 + \ldots + \beta_n\psi_n)$ and $\phi_2 = (\gamma_2\psi_2 + \ldots + \gamma_n\psi_n)$ to be linearly independent vectors satisfying the system of equations $\beta_1\gamma_j = \alpha_{1j}$ for $j \geq 2$ and $\beta_i\gamma_2 = -\alpha_{i2}$ for $i \geq 3$ (such vectors exist for $n \geq 3$

because there are at least more variables than unknowns, and the case for $n \le 3$ is proved in question 7-6 in the last problem set). Then we see that $\omega - \phi_1 \wedge \phi_2 = \sum_{i<j} \alpha_{ij}\psi_i \wedge \psi_j - (\beta_1\psi_1 \wedge \phi_2 - \gamma_2\psi_2 \wedge (\phi_1 - \beta_1\psi_1) + \eta) = \sum_{i<j, i,j \notin \{1,2\}} \alpha_{ij}\psi_i \wedge \psi_j + \eta$ where η is the part of $\phi_1 \wedge \phi_2$ that does not involve ψ_1, ψ_2, and hence letting $\omega' = \sum_{i<j,i,j\notin\{1,2\}} \alpha_{ij}\psi_i \wedge \psi_j + \eta$ we see that $\omega = \phi_1 \wedge \phi_2 + \omega'$ where ω' does not involve ψ_1 or ψ_2. Now using induction on n and the "extending to a basis" theorem in linear algebra, we conclude that there is a basis $\phi_1, ..., \phi_n$ of V^* such that $\omega = (\phi_1 \wedge \phi_2) + ... + (\phi_{2r-1} \wedge \phi_{2r})$ for some r.

(b) Suppose we write $\omega = (\phi_1 \wedge \phi_2) + ... + (\phi_{2r-1} \wedge \phi_{2r})$ as constructed in part (a). Now consider $\omega \wedge ... \wedge \omega$ for r-times, we see that when we expand out the whole expression, the terms in $\omega^{\wedge r}$ that do not involve the same ϕ_i (i.e. the only nonzero terms) are that of the form $(\phi_1 \wedge \phi_2 \wedge ... \wedge \phi_{2r-1} \wedge \phi_{2r})$. To make this precise, write $\omega = (\phi_1 \wedge \phi_2) + ... + (\phi_{2r-1} \wedge \phi_{2r}) = A_1 + ... + A_r$, then

$$\omega^{\wedge r} = \sum_{(p_1,...,p_r) \in \{1,...,r\}^{\times r}} A_{p_1} \wedge ... \wedge A_{p_r}$$

and the only terms in the above summand that is nonzero is when $(p_1, ..., p_r)$ has no repeated entry (because if for some term $p_i = p_j$ then that term will have repeated ϕ_k for some k, and hence the wedge product will be 0), and so the above simplifies to

$$\omega^{\wedge r} = \sum_{\sigma \in S_r} A_{\sigma(1)} \wedge ... \wedge A_{\sigma(r)} = r! \cdot A_1 \wedge ... \wedge A_r = r! \cdot \phi_1 \wedge \phi_2 \wedge ... \wedge \phi_{2r-1} \wedge \phi_{2r}$$

which is non-zero and decomposible.

On the other hand, consider $\omega^{\wedge r+1}$, we have

$$\omega^{\wedge r+1} = \sum_{(p_1,...,p_{r+1}) \in \{1,...,r\}^{\times r+1}} A_{p_1} \wedge ... \wedge A_{p_{r+1}}$$

and since we are choosing $r+1$ things from the set $\{1, ..., r\}$ of r elements, the expression $(p_1, ..., p_{r+1})$ must have repeated p_i in the entries, and hence $A_{p_1} \wedge ... \wedge A_{p_{r+1}} = 0$ because it will have repeated ϕ_i in it. Thus we see that all the summand is actually zero, and hence

$$\omega^{\wedge r+1} = \sum_{(p_1,...,p_{r+1}) \in \{1,...,r\}^{\times r+1}} A_{p_1} \wedge ... \wedge A_{p_{r+1}} = 0$$

Thus the number r is well-defined.

Question 4.

Let the v_i defined in the problem be denoted as $v_i = \sum_{j=1}^n \gamma_{ij} e_j$ where e_j is the standard basis for $\mathbb{R}^n$. We let $G : [0, \epsilon]^n \to P_\epsilon$ be the map taking $(x_1, ..., x_n) \mapsto (\gamma_{11}x_1 + ... + \gamma_{1n}x_n, ..., \gamma_{n1}x_1 + ... + \gamma_{nn}x_n)$. Note that in euclidean space, from multivariable calculus we know that there is a "mean value" property for integrals of continuous functions, i.e.

$$\int_S f(x)dx = f(c) \int_S 1dx$$

where $x = (x^1, ..., x^n)$ and S a measurable set and $c \in S$. So now it follows

$$\begin{aligned}
\int_{\partial P_\epsilon} \omega &= \int_{P_\epsilon} d\omega \\
&= \int_{[0,\epsilon]^n} G^* d\omega \\
&= (G^* d\omega)|_c \left(\left.\frac{\partial}{\partial x^1}\right|_c, ..., \left.\frac{\partial}{\partial x^n}\right|_c \right) \int_{[0,\epsilon]^n} 1dx \\
&= d\omega|_{G(c)} \left(G_* \left.\frac{\partial}{\partial x^1}\right|_c, ..., G_* \left.\frac{\partial}{\partial x^n}\right|_c \right) \epsilon^n
\end{aligned}$$

where $c \in [0,\epsilon]^n$. Thus we see that

$$\begin{aligned}
\lim_{\epsilon\to 0}\frac{1}{\epsilon^n}\int_{\partial P_\epsilon}\omega &= \lim_{\epsilon\to 0}\frac{1}{\epsilon^n}d\omega|_{G(c)}\left(G_*\frac{\partial}{\partial x^1}\bigg|_c,...,G_*\frac{\partial}{\partial x^n}\bigg|_c\right)\epsilon^n \\
&= \lim_{\epsilon\to 0}d\omega|_{G(c)}\left(G_*\frac{\partial}{\partial x^1}\bigg|_c,...,G_*\frac{\partial}{\partial x^n}\bigg|_c\right) \\
&= \lim_{\epsilon\to 0}d\omega|_{G(c)}\left(\sum_{j=1}^n\gamma_{1j}\frac{\partial}{\partial x^j}\bigg|_{G(c)},...,\sum_{j=1}^n\gamma_{nj}\frac{\partial}{\partial x^j}\bigg|_{G(c)}\right) \\
&= d\omega|_{G(0)}\left(\sum_{j=1}^n\gamma_{1j}\frac{\partial}{\partial x^j}\bigg|_{G(0)},...,\sum_{j=1}^n\gamma_{nj}\frac{\partial}{\partial x^j}\bigg|_{G(0)}\right) \\
&= d\omega|_0(v_1,...,v_n)
\end{aligned}$$

Question 7-18.

(a) Suppose ω is a k-form, then this is equivalent as saying $\omega_p(X_1,...,Y,...,Y,...,X_k) = 0\forall p$, i.e. that for all p, ω is zero whenever any two of its arguments is the same. Now $(L_X\omega)_p = \lim_{h\to 0}\frac{1}{h}[(\phi_h^*\omega)_p - \omega_p]$, so $(L_X\omega)_p(X_1,...,Y,...,Y,...,X_k) = \lim_{h\to 0}\frac{1}{h}[(\phi_h^*\omega)_p(X_1,...,Y,...,Y,...,X_k) - \omega_p(X_1,...,Y,...,Y,...,X_k)] = \lim_{h\to 0}\frac{1}{h}[0-0] = 0$, and hence $L_X\omega$ is also a k-form.

(b)

$$\begin{aligned}
L_X(\omega\wedge\eta) &= L_X\left(\frac{(k+l)!}{k!l!}\text{Alt}(\omega\otimes\eta)\right) = \frac{(k+l)!}{k!l!}L_X\left(\frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}}(\text{sgn}\sigma)(\omega\otimes\eta)^\sigma\right) \\
&= \frac{(k+l)!}{k!l!}\left(\frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}}(\text{sgn}\sigma)L_X(\omega\otimes\eta)^\sigma\right) \\
&= \frac{(k+l)!}{k!l!}\left(\frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}}(\text{sgn}\sigma)((L_X\omega\otimes\eta)^\sigma + (\omega\otimes L_X\eta)^\sigma)\right) \\
&= \frac{(k+l)!}{k!l!}\left(\frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}}(\text{sgn}\sigma)((L_X\omega\otimes\eta)^\sigma\right) + \frac{(k+l)!}{k!l!}\left(\frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}}(\text{sgn}\sigma)(\omega\otimes L_X\eta)^\sigma\right) \\
&= \frac{(k+l)!}{k!l!}\left(\text{Alt}(L_X\omega\otimes\eta)\right) + \frac{(k+l)!}{k!l!}\left(\text{Alt}(\omega\otimes L_X\eta)\right) \\
&= L_X\omega\wedge\eta + \omega\wedge L_X\eta
\end{aligned}$$

where the second line to the third line follows from 5-14 part (b) in Spivak.

(c) Problem 5-14 (e) in Spivak tells us that

$$\begin{aligned}
L_X(A(X_1,...,X_k,\omega_1,...,\omega_l)) &= (L_XA)(X_1,...,X_k,\omega_1,...,\omega_l) \\
&\quad + \sum_{i=1}^k A(X_1,...,L_XX_i,...,X_k,\omega_1,...,\omega_l) + \sum_{i=1}^l A(X_1,...,X_k,\omega_1,...,L_X\omega_i,...,\omega_l)
\end{aligned}$$

So applying this to our problem, we see that

$$\begin{aligned} X(\omega(X_1,...,X_k)) &= L_X(\omega(X_1,...,X_k)) \\ &= (L_X\omega)(X_1,...,X_k) + \sum_{i=1}^{k} \omega(X_1,...,L_X X_i,...,X_k) \\ &= (L_X\omega)(X_1,...,X_k) + \sum_{i=1}^{k} \omega(X_1,...,[X,X_i],...,X_k) \\ &= (L_X\omega)(X_1,...,X_k) + \sum_{i=1}^{k} (-1)^{i-1}\omega([X,X_i],X_1,...,\hat{X}_i,...,X_k) \end{aligned}$$

(d) In Theorem 7-13 in Spivak, it is shown that

$$d\omega(X_1,...,X_{k+1}) = \sum_{i=1}^{k+1}(-1)^{i+1}X_i(\omega(X_1,...,\hat{X}_i,...,X_{k+1})) + \sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_1,...,\hat{X}_i,...,\hat{X}_j,...,X_{k+1})$$

so using part (c), we see that

$$\begin{aligned} d\omega(X_1,...,X_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i+1}X_i(\omega(X_1,...,\hat{X}_i,...,X_{k+1})) + \sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_1,...,\hat{X}_i,...,\hat{X}_j,...,X_{k+1}) \\ &= \sum_{i=1}^{k+1}(-1)^{i+1}(L_{X_i}\omega)(X_1,...,\hat{X}_i,...,X_{k+1}) + \sum_{i=1}^{k+1}(-1)^{i+1}\sum_{j<i}(-1)^{j+1}\omega([X_i,X_j],X_1,...,\hat{X}_i,...,\hat{X}_j,...,X_{k+1}) \\ &+ \sum_{i=1}^{k+1}(-1)^{i+1}\sum_{j>i}(-1)^{j}\omega([X_i,X_j],X_1,...,\hat{X}_i,...,\hat{X}_j,...,X_{k+1}) \\ &+ \sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_1,...,\hat{X}_i,...,\hat{X}_j,...,X_{k+1}) \\ &= \sum_{i=1}^{k+1}(-1)^{i+1}(L_{X_i}\omega)(X_1,...,\hat{X}_i,...,X_{k+1}) \\ &+ 2\sum_{j>i}(-1)^{i+j+1}\omega([X_i,X_j],X_1,...,\hat{X}_i,...,\hat{X}_j,...,X_{k+1}) + \sum_{j>i}(-1)^{i+j}\omega([X_i,X_j],X_1,...,\hat{X}_i,...,\hat{X}_j,...,X_{k+1}) \\ &= \sum_{i=1}^{k+1}(-1)^{i+1}(L_{X_i}\omega)(X_1,...,\hat{X}_i,...,X_{k+1}) + \sum_{j>i}(-1)^{i+j+1}\omega([X_i,X_j],X_1,...,\hat{X}_i,...,\hat{X}_j,...,X_{k+1}) \end{aligned}$$

(e) By part (d), we see that $d\omega(X_1,...,X_{k+1}) = L_{X_1}\omega(X_2,...,X_{k+1}) + \sum_{i=2}^{k+1}(-1)^{i+1}(L_{X_i}\omega)(X_1,...,\hat{X}_i,...,X_{k+1}) + \sum_{j>i}(-1)^{i+j+1}\omega([X_i,X_j],X_1,...,\hat{X}_i,...,\hat{X}_j,...,X_{k+1})$, and applying part (d) again to $(X_1 \lrcorner \omega)$ gives

$$\begin{aligned} d(X_1 \lrcorner \omega)(X_2,...,X_{k+1}) \\ &= \sum_{i=2}^{k+1}(-1)^{i}(L_{X_i}(X_1\lrcorner\omega))(X_2,...,\hat{X}_i,...,X_{k+1}) + \sum_{j>i}(-1)^{i+j+1}(X_1\lrcorner\omega)([X_i,X_j],X_2,...,\hat{X}_i,...,\hat{X}_j,...,X_{k+1}) \\ &= \sum_{i=2}^{k+1}(-1)^{i}(L_{X_i}\omega)(X_1,X_2,...,\hat{X}_i,...,X_{k+1}) + \sum_{j>i}(-1)^{i+j+1}\omega(X_1,[X_i,X_j],X_2,...,\hat{X}_i,...,\hat{X}_j,...,X_{k+1}) \\ &= \sum_{i=2}^{k+1}(-1)^{i}(L_{X_i}\omega)(X_1,X_2,...,\hat{X}_i,...,X_{k+1}) + \sum_{j>i}(-1)^{i+j}\omega([X_i,X_j],X_1,X_2,...,\hat{X}_i,...,\hat{X}_j,...,X_{k+1}) \end{aligned}$$

Note that this expression is exactly the last two terms in the equation for $d\omega$ at the beginning of this part (e). Thus we see that

$$d\omega(X_1,...,X_{k+1}) = L_{X_1}\omega(X_2,...,X_{k+1}) - d(X_1\lrcorner\omega)(X_2,...,X_{k+1})$$

and hence

$$X \lrcorner d\omega = L_X\omega - d(X \lrcorner \omega)$$

(f) Plugging in $\omega = d\eta$ in part (e) gives $0 = X \lrcorner dd\eta = L_X d\eta - d(X \lrcorner d\eta)$, and so $L_X d\eta = d(X \lrcorner d\eta)$. On the other hand, taking d on both sides of part (e) gives $d(X \lrcorner d\eta) = d(L_X\eta - d(X \lrcorner \eta) = d(L_X\eta) - 0$, and hence $d(X \lrcorner d\eta) = d(L_X\eta)$. Combining the two equations we conclude that

$$L_X d\eta = d(X \lrcorner d\eta) = d(L_X\eta)$$

Question 8-28.

(a) Let ω_0 be an $(n-1)$-form on S^{n-1} such that $\int_{S^{n-1}} \omega_0 \neq 0$, then the degree of $\alpha_{f,g}$ is defined to be

$$l(f,g) = \deg \alpha_{f,g} = \frac{\int_{M\times N} \alpha_{f,g}^* \omega_0}{\int_{S^{n-1}} \omega_0}$$

We let $H : M \times N \to N \times M$ to be the map that take $(p,q) \mapsto (q,p)$. H^* has degree $(-1)^{kl}$ because in coordinates $(m_1, ..., m_k)$ for M and $(n_1, ..., n_l)$ for N, $(dm_1 \wedge ... \wedge dm_k) \wedge (dn_1 \wedge ... \wedge dn_l) = (-1)^{kl}(dn_1 \wedge ... \wedge dn_l) \wedge (dm_1 \wedge ... \wedge dm_k)$ By definition, $\alpha_{f,g}(p,q) = (-1)\alpha_{g,f}(q,p)$, and so we have

$$\begin{aligned}\int_{M\times N} \alpha_{f,g}^* \omega_0 &= \int_{N\times M} (-1)^{kl} H^* \alpha_{f,g}^* \omega_0 = \int_{N\times M} (-1)^{kl} (\alpha_{f,g} \circ H)^* \omega_0 \\ &= \int_{N\times M} (-1)^{kl} (-1 \cdot \alpha_{g,f})^* \omega_0 = \int_{N\times M} (-1)^{kl+1} (\alpha_{g,f})^* \omega_0.\end{aligned}$$

and thus we see that

$$l(f,g) = \deg \alpha_{f,g} = \frac{\int_{M\times N} \alpha_{f,g}^* \omega_0}{\int_{S^{n-1}} \omega_0} = (-1)^{kl+1} \frac{\int_{N\times M} (-1)^{kl+1} (\alpha_{g,f})^* \omega_0}{\int_{S^{n-1}} \omega_0} = (-1)^{kl+1} \deg \alpha_{g,f} = (-1)^{kl+1} l(g,f)$$

(b) Define $\Gamma : M \times N \times [0,1] \to S^{n-1} \subset \mathbb{R}^n - \{0\}$ by taking $(p,q,t) \mapsto \frac{K(q,t)-H(p,t)}{|K(q,t)-H(p,t)|}$ (this is well defined because of the condition $\{H(p,t) : p \in M\} \cap \{K(q,t) : q \in N\} = \varnothing$ for all t). This defines a smooth map with $\Gamma(p,q,0) = \alpha_{f,g}(p,q)$ and $\Gamma(p,q,1) = \alpha_{\bar{f},\bar{g}}(p,q)$ and hence it is a smooth homotopy between $\alpha_{\bar{f},\bar{g}}$ and $\alpha_{f,g}(p,q)$. Since homotopic maps have the same degree (as they induce the same map on cohomology), we see that $l(f,g) = \deg \alpha_{f,g} = \deg \alpha_{\bar{f},\bar{g}} = l(\bar{f},\bar{g})$.

(c) We pick σ' to be the 2-form on S^2 that is defined on p264-265 of Spivak, and let $r^*\sigma' = \frac{xdy\wedge dz - ydx\wedge dz + zdx\wedge dy}{(x^2+y^2+z^2)^{3/2}}$ be the form on $\mathbb{R}^3$ defined on p265 of Spivak, where r^* has degree 1 because it is a retraction. Then

$$\begin{aligned}
l(f,g) &= \deg \alpha_{f,g} = \deg(r \circ \alpha_{f,g}) = \frac{\int_{S^1\times S^1} \alpha_{f,g}^* r^* \sigma'}{\int_{S^2} \sigma'} = \frac{1}{4\pi} \int_{S^1\times S^1} \alpha_{f,g}^* r^* \sigma' \\
&= \frac{1}{4\pi} \int_{S^1\times S^1} \alpha_{f,g}^* \frac{xdy\wedge dz - ydx\wedge dz + zdx\wedge dy}{(x^2+y^2+z^2)^{3/2}} \\
&= \frac{1}{4\pi} \int_{S^1\times S^1} \frac{1}{(x^2+y^2+z^2)^{3/2} \circ \alpha_{f,g}} (x \circ \alpha_{f,g} d(y \circ \alpha_{f,g}) \wedge d(z \circ \alpha_{f,g}) \\
&- y \circ \alpha_{f,g} d(x \circ \alpha_{f,g}) \wedge d(z \circ \alpha_{f,g}) + z \circ \alpha_{f,g} d(x \circ \alpha_{f,g}) \wedge d(y \circ \alpha_{f,g})) \\
&= \frac{1}{4\pi} \int_0^1 \int_0^1 \frac{g^1(v) - f^1(u)}{|g(v) - f(u)|} \left(\frac{(g^2)'(v)dv - (f^2)'(u)du}{|g(v) - f(u)|} \right) \wedge \left(\frac{(g^3)'(v)dv - (f^3)'(u)du}{|g(v) - f(u)|} \right) \\
&- \frac{g^2(v) - f^2(u)}{|g(v) - f(u)|} \left(\frac{(g^1)'(v)dv - (f^1)'(u)du}{|g(v) - f(u)|} \right) \wedge \left(\frac{(g^3)'(v)dv - (f^3)'(u)du}{|g(v) - f(u)|} \right) \\
&+ \frac{g^3(v) - f^3(u)}{|g(v) - f(u)|} \left(\frac{(g^1)'(v)dv - (f^1)'(u)du}{|g(v) - f(u)|} \right) \wedge \left(\frac{(g^2)'(v)dv - (f^2)'(u)du}{|g(v) - f(u)|} \right) \\
&= \frac{1}{4\pi} \int_0^1 \int_0^1 \frac{(g^1(v) - f^1(u))(-(f^2)'(u)(g^3)'(u)du \wedge dv + (f^3)'(u)(g^2)'(u)du \wedge dv)}{|g(v) - f(u)|^3} \\
&- \frac{1}{4\pi} \int_0^1 \int_0^1 \frac{(g^2(v) - f^2(u))(-(f^1)'(u)(g^3)'(u)du \wedge dv + (f^3)'(u)(g^1)'(u)du \wedge dv)}{|g(v) - f(u)|^3} \\
&+ \frac{1}{4\pi} \int_0^1 \int_0^1 \frac{(g^3(v) - f^3(u))(-(f^1)'(u)(g^2)'(u)du \wedge dv + (f^2)'(u)(g^1)'(u)du \wedge dv)}{|g(v) - f(u)|^3} \\
&= \frac{-1}{4\pi} \int_0^1 \int_0^1 \frac{A(u,v)}{r(u,v)^3} du \wedge dv
\end{aligned}$$

Where $r(u,v) = |g(v) - f(u)|$ and

$$A(u,v) = \det\begin{pmatrix} (f^1)'(u) & (f^2)'(u) & (f^3)'(u) \\ (g^1)'(v) & (g^2)'(v) & (g^3)'(v) \\ g^1(v) - f^1(u) & g^2(v) - f^2(u) & g^3(v) - f^3(u) \end{pmatrix}$$

(d) If f and g both lie in the xy-plane, then the matrix in part (c) has the last column all zero, and hence the determinant is zero, which means that $A(u,v) = 0$ for all $(u,v) \in [0,1]^2$. Thus we see that if f and g both lie in the xy-plane, then $l(f,g) = 0$. Now if f and g lies in the same plane, then we see that the matrix A for this f,g differs by the matrix A for the xy-plane case by a rotation, and hence is still singular, with determinant 0. Thus if f and g lies in the same plane, then $l(f,g) = 0$.

Question 8-29.

(a) Suppose $M = \partial N$. If $(a,b,c) \in N - M$, then let $B_{(a,b,c)}$ be a ball centerted at (a,b,c) and completely contained in $N - M$. Then by question 14 which is in the previous homework, we see that

$$\Omega(a,b,c) = \int_M d\Theta_{(a,b,c)} = \int_{\partial N} d\Theta_{(a,b,c)} = \int_{\partial B_{(a,b,c)}} d\Theta_{(a,b,c)} = \int_{S^2} \sigma' = -4\pi$$

On the other hand, if $(a,b,c) \notin N$, then

$$\Omega(a,b,c) = \int_M d\Theta_{(a,b,c)} = \int_{\partial N} d\Theta_{(a,b,c)} = \int_N dd\Theta_{(a,b,c)} = 0$$

So back to the problem, in the limit that $(a,b,c),(a',b',c') \to p$, we can approximate a neighborhood $U \subset M$ of p to be approximately ∂N where N is the half space. Thus in the orientation defined in the problem, we see that

$$\lim_{(a,b,c),(a',b',c')\to p} \Omega(a,b,c) - \Omega(a',b',c') = -4\pi$$

(b) For any smooth curve $\gamma : [p,q] \to \mathbb{R}^3$ which is an embedding such that the image is contained in $\mathbb{R}^3 - M$, we see that $\int_\gamma d\Omega = \Omega(\gamma(q)) - \Omega(\gamma(p)) = 0$ by part (a). On the other hand, if γ pass through M exactly once at time t_0, and $\frac{dg}{dt}|_{t=t_0}$ is in the direction of w_p, then $\int_\gamma d\Omega = \lim_{\epsilon\to 0} \Omega(g(t_0+\epsilon)) - \Omega(g(t_0-\epsilon)) = -4\pi$ by part (a) again, while if $\frac{dg}{dt}|_{t=t_0}$ is in the opposite direction of w_p, then $\int_\gamma d\Omega = \lim_{\epsilon\to 0} \Omega(g(t_0+\epsilon)) - \Omega(g(t_0-\epsilon)) = +4\pi$. Since $g(S^1)$ is just a closed curve in $\mathbb{R}^3$, by these observations above, we see that

$$\int_{S^1} g^*(d\Omega) = \int_{g(S^1)} d\Omega = -4\pi(n^+ - n^-)$$

and thus

$$n = n^+ - n^- = \frac{-1}{4\pi} \int_{S^1} g^*(d\Omega)$$

(c) From this point on, $|(x,y,z)| = ((x-a)^2 + (y-b)^2 + (z-c)^2)^{1/2}$.

$$\int_{S^1} f^* \left(\frac{(y-b)dz - (z-c)dy}{|x,y,z|^3} \right) = \int_{f(S^1)} \left(\frac{(y-b)dz - (z-c)dy}{|x,y,z|^3} \right) = \int_{\partial M} \left(\frac{(y-b)dz - (z-c)dy}{|x,y,z|^3} \right)$$

$$= \int_M d\left(\frac{(y-b)dz - (z-c)dy}{|(x,y,z)|^3} \right)$$

$$= \int_M \left(\frac{-3(x-a)(y-b)}{|(x,y,z)|^5} dx \wedge dz + \left(\frac{1}{|(x,y,z)|^3} - \frac{3(y-b)^2}{|(x,y,z)|^5} \right) dy \wedge dz \right)$$

$$- \int_M \left(\frac{-3(x-a)(z-c)}{|(x,y,z)|^5} dx \wedge dy + \left(\frac{1}{|(x,y,z)|^3} - \frac{3(z-c)^2}{|(x,y,z)|^5} \right) dz \wedge dy \right)$$

$$= \int_M \left(\frac{2}{|(x,y,z)|^3} - \frac{3(y-b)^2 + 3(z-c)^2}{|(x,y,z)|^5} dz \wedge dy \right)$$

$$+ \int_M \left(\frac{-3(x-a)(y-b)}{|(x,y,z)|^5} dx \wedge dz + \frac{3(x-a)(z-c)}{|(x,y,z)|^5} dx \wedge dy \right)$$

$$= \int_M \left(\frac{2}{|(x,y,z)|^3} - \frac{3(y-b)^2 + 3(z-c)^2 + 3(x-a)^2}{|(x,y,z)|^5} + \frac{3(x-a)^2}{|(x,y,z)|^5} dz \wedge dy \right)$$

$$+ \int_M \left(\frac{-3(x-a)(y-b)}{|(x,y,z)|^5} dx \wedge dz + \frac{3(x-a)(z-c)}{|(x,y,z)|^5} dx \wedge dy \right)$$

$$= \int_M \left(\frac{-1}{|(x,y,z)|^3} + \frac{3(x-a)^2}{|(x,y,z)|^5} dz \wedge dy \right) + \int_M \left(\frac{-3(x-a)(y-b)}{|(x,y,z)|^5} dx \wedge dz + \frac{3(x-a)(z-c)}{|(x,y,z)|^5} dx \wedge dy \right)$$

$$= \frac{\partial}{\partial a}\bigg|_{(a,b,c)} \int_M \frac{(x-a)dy \wedge dz - (y-b)dx \wedge dz + (z-c)dx \wedge dy}{|(x,y,z)|^3} = \frac{\partial \Omega}{\partial a}(a,b,c)$$

The calculation for $\frac{\partial}{\partial b}\Omega(a,b,c)$ and $\frac{\partial}{\partial c}\Omega(a,b,c)$ is completely analogous.

(d)

$$n = \frac{-1}{4\pi} \int_{S^1} g^*(d\Omega)$$

$$= \frac{-1}{4\pi} \int_{S^1} g^* \left(\frac{\partial}{\partial a} da + \frac{\partial}{\partial b} db + \frac{\partial}{\partial c} dc \right)$$

$$= \frac{-1}{4\pi} \int_{S^1} g^* \int_{S^1} f^* \left(\frac{(y-b)dz - (z-c)dy}{|x,y,z|^3} \right) da + f^* \left(\frac{(z-c)dx - (x-a)dz}{|x,y,z|^3} \right) db + f^* \left(\frac{(x-a)dy - (y-b)dx}{|x,y,z|^3} \right) dc$$

$$= \frac{-1}{4\pi} \int_{S^1} \int_{u=0}^1 g^* \left(\frac{(f^2(u) - b)df^3(u) - (f^3(u) - c)df^2(u)}{|f^1(u), f^2(u), f^(3)(u)|^3} \right) da$$

$$+ g^* \left(\frac{(f^3(t) - c)df^1(u) - (f^1(u) - a)df^3(t)}{|f^1(u), f^2(u), f^(3)(u)|^3} \right) db + g^* \left(\frac{(f^1(u) - a)df^2(u) - (f^2(u) - b)df^1(u)}{|f^1(u), f^2(u), f^(3)(u)|^3} \right) dc$$

$$= \frac{-1}{4\pi} \int_{v=0}^1 \int_{u=0}^1 \left(\frac{(f^2(u) - g^2(v))df^3(u) - (f^3(u) - g^3(u))df^2(u)}{|r(u,v)|^3} \right) \wedge dg^1(v)$$

$$+ \left(\frac{(f^3(t) - g^3(v))df^1(u) - (f^1(u) - g^1(v))df^3(t)}{|r(u,v)|^3} \right) \wedge dg^2(v)$$

$$+ \left(\frac{(f^1(u) - g^1(v))df^2(u) - (f^2(u) - g^2(v))df^1(u)}{|r(u,v)|^3} \right) \wedge dg^3(v)$$

$$= \frac{-1}{4\pi} \int_0^1 \int_0^1 \frac{A(u,v)}{|r(u,v)|^3} du \wedge dv$$

$$= l(f,g)$$

where $r(u,v)$ and $A(u,v)$ are defined as in problem 8-28 part (c) above. Thus we see that

$$n = n^+ - n^- = l(f,g)$$

For the three pictures in the book, the first picture from the left has $|l(f,g)| = 1$, the second picture from the left has $|l(f,g)| = 2$, and the third picture from the left has $|l(f,g)| = 0$ by counting $n^+ - n^-$.