Math 214 Problem Set 5

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Question 6-6.

Theorem 5 states that every C^∞ integrable k-dimensional distribution Δ on M has an integral manifold. So consider the distribution Δ in $\mathbb{R}^m \times \mathbb{R}^n$ defined by

$$\Delta_p = \left\{ \sum_{i=1}^m r^i \frac{\partial}{\partial t^i}\bigg|_p + \sum_{k=1}^n \left(\sum_{i=1}^m r^i f_i^k(p) \right) \frac{\partial}{\partial x^k}\bigg|_p : r \in \mathbb{R}^m \right\}$$

Note that Δ is spanned by the vector fields

$$X_i = \frac{\partial}{\partial t_i} + \sum_{k=1}^n f_i^k \frac{\partial}{\partial x^k} \text{ , for } i = 1, ..., m$$

We will first show that Δ is a C^∞ integrable k-dimensional distribution if and only if condition $(**)$ of Theorem 1 is satisfied, and hence Δ has an integral manifold if and only if condition $(**)$ is satisfied. Let our integral manifold be a graph of a function $\alpha(r_1, ..., r_m) = (\alpha^1(r_1, ..., r_m), ..., \alpha^n(r_1, ..., r_m))$, i.e. the integral manifold looks like $\{(r_1, ..., r_m, \alpha^1(r_1, ..., r_m), ..., \alpha^n(r_1, ..., r_m)) : r = (r_1, ..., r_m) \in \mathbb{R}^n\}$, then being an integral manifold of Δ translates to the conditions $\frac{\partial}{\partial t_i} + \frac{\partial \alpha^1}{\partial t_i}\frac{\partial}{\partial x^1} + ... + \frac{\partial \alpha^n}{\partial t_i}\frac{\partial}{\partial x^n} \in \Delta_p$, which means $\frac{\partial \alpha^k}{\partial t_i} = f_i^k$, and so $\frac{\partial \alpha}{\partial t_i} = f_i$, which are precisely the differential equations that Theorem 1 solves. Thus we only need to show that Δ is a C^∞ integrable k-dimensional distribution if and only if condition $(**)$ of Theorem 1 is satisfied (which, together with the argument above, will prove Theorem 1).

We will verify that Δ is a C^∞ integrable k-dimensional distribution if and only if condition $(**)$ of Theorem 1 is satisfied. Now by proposition 4 we only need to check that the Lie bracket on the X_i belongs to Δ. We now compute

$$[X_i, X_j] = \sum_{k=1}^n \left(\frac{\partial f_j^k}{\partial t^i} - \frac{\partial f_i^k}{\partial t^j} + \sum_{k'=1}^n f_i^{k'} \frac{\partial f_j^k}{\partial x^{k'}} - \sum_{k'=1}^n f_j^{k'} \frac{\partial f_i^k}{\partial x^{k'}} \right) \cdot \frac{\partial}{\partial x^k}$$

Thus $[X_i, X_j] \in \Delta$ if and only if

$$\frac{\partial f_j^k}{\partial t^i} - \frac{\partial f_i^k}{\partial t^j} + \sum_{k'=1}^n f_i^{k'} \frac{\partial f_j^k}{\partial x^{k'}} - \sum_{k'=1}^n f_j^{k'} \frac{\partial f_i^k}{\partial x^{k'}} = 0 \text{ for all } k = 1, ..., n$$

and as i, j are arbitrary, this is precisely condition $(**)$, and we have proved Theorem 1 from Theorem 5.

Question 6-8.

Define $\alpha(x,y) = (\alpha^1(x,y), \alpha^2(x,y))$, and letting $f_1 = (u(x, y, \alpha^1(x,y), \alpha^2(x,y)), v(x, y, \alpha^1(x,y), \alpha^2(x,y)))$ and $f_2 = (-v(x, y, \alpha^1(x,y), \alpha^2(x,y)), u(x, y, \alpha^1(x,y), \alpha^2(x,y)))$, the differential equation listed in the problem translates to solving $\frac{\partial \alpha}{\partial x} = f_1$ and $\frac{\partial \alpha}{\partial y} = f_2$. Now Theorem 1 tells us when this system of differential equations have a solution: when f_1, f_2 satisfies condition $(**)$. Thus it remains to check condition $(**)$. Since $i,j = 1,2$ in our case, we see that we only need to check $i = 1, j = 2$ (because condition $(**)$ is automatically satisfied for $i = j$ and the case $i = 2, j = 1$ is just -1 times the case $i = 1, j = 2$). Now

$$\begin{aligned}(**) &= \frac{\partial f_1}{\partial y^1} - \frac{\partial f_2}{\partial x^1} + \frac{\partial f_1}{\partial x^2} f_2^1 + \frac{\partial f_1}{\partial y^2} f_2^2 - \frac{\partial f_2}{\partial x^2} f_1^1 - \frac{\partial f_2}{\partial y^2} f_2^2 \\ &= \left(\frac{\partial u}{\partial y^1}, \frac{\partial v}{\partial y^1}\right) - \left(-\frac{\partial v}{\partial x^1}, \frac{\partial u}{\partial x^1}\right) + \left(\frac{\partial u}{\partial x^2}, \frac{\partial v}{\partial x^2}\right)(-v) + \left(\frac{\partial u}{\partial y^2}, \frac{\partial v}{\partial y^2}\right)(u) - \left(\frac{\partial - v}{\partial x^2}, \frac{\partial u}{\partial x^2}\right)(u) - \left(\frac{\partial - v}{\partial y^2}, \frac{\partial u}{\partial y^2}\right)(v) \\ &= 0\end{aligned}$$

Where the last equality follows from the Cauchy-Riemann relations of u, v. Thus $(**)$ is satisfied, and hence α is the solution to the differential equations listed above. Therefore (sorry, I am changing notations to $\alpha = \phi$) we conclude that the differential equation $\phi'(z) = f(z, \phi(z))$ has a solution in a neighborhood of z_0 with initial condition $\phi(z_0) = w_0$.

Question 7-2.

We want to show that $\bar{\wedge}$ is not associative. To see this, we let $\omega, \eta \in \Omega^1(V)$ and $\theta \in \Omega^2(V)$. Then

$$\begin{aligned}\omega\bar{\wedge}(\eta\bar{\wedge}\theta)(v_1, v_2, v_3, v_4) &= w\bar{\wedge}\left(\sum_{\sigma\in S_3} sgn(\sigma)(\eta\otimes\theta)^\sigma\right)(v_1, v_2, v_3, v_4) \\ &= \sum_{\xi\in S_4} sgn(\xi)\left(\omega\otimes\sum_{\sigma\in S_3} sgn(\sigma)(\eta\otimes\theta)^\sigma\right)^\xi (v_1, v_2, v_3, v_4) \\ &= \sum_{\xi\in S_4} sgn(\xi)\sum_{\sigma\in S_3} sgn(\sigma)\omega(v_{\xi(1)})\eta(v_{\sigma\circ\xi(2)})\theta(v_{\sigma\circ\xi(3)}, v_{\sigma\circ\xi(4)})\end{aligned}$$

On the other hand

$$\begin{aligned}(\omega\bar{\wedge}\eta)\bar{\wedge}\theta(v_1, v_2, v_3, v_4) &= \left(\sum_{\sigma\in S_2} sgn(\sigma)(\omega\otimes\eta)^\sigma\right)\bar{\wedge}\theta(v_1, v_2, v_3, v_4) \\ &= \sum_{\xi\in S_4} sgn(\xi)\left(\sum_{\sigma\in S_2} sgn(\sigma)(\omega\otimes\eta)^\sigma\otimes\theta\right)^\xi (v_1, v_2, v_3, v_4) \\ &= \sum_{\xi\in S_4} sgn(\xi)\sum_{\sigma\in S_2} sgn(\sigma)\omega(v_{\sigma\circ\xi(1)})\eta(v_{\sigma\circ\xi(2)})\theta(v_{\xi(3)}, v_{\xi(4)})\end{aligned}$$

The two expressions are not equal, because the first one sums over S_4 "three times", while the second one sums over S_4 "two times". Thus we see that the $\bar{\wedge}$ so defined is NOT associative.

Question 7-4.

(a) Let $a, b \in V$ and $c \in \Omega^k(V)$, then for any $v_1, ..., v_{k-2}$

$$\begin{aligned}a\lrcorner(b\lrcorner c)(v_1, ..., v_{k-2}) &= b\lrcorner c(a, v_1, ..., v_{k-2}) \\ &= c(b, a, v_1, ..., v_{k-2}) \\ &= -c(a, b, v_1, ..., v_{k-2}) \\ &= -a\lrcorner c(b, v_1, ..., v_{k-2}) \\ &= -b\lrcorner(a\lrcorner c)(v_1, ..., v_{k-2})\end{aligned}$$

Thus $a\lrcorner(b\lrcorner c) = -b\lrcorner(a\lrcorner c)$.

(b) Let $v_1, ..., v_n$ be a basis of V and $\phi_1, ..., \phi_n$ be the dual basis, then letting $v_j = X_1$ we see that $v_j \lrcorner (\phi_{i_1} \wedge ... \wedge \phi_{i_k})(X_2, ..., X_k) = \phi_{i_1} \wedge ... \wedge \phi_{i_k}(X_1, ..., X_k) = \det(\phi_{i_r}(X_s))$. If $j \neq i_\alpha$ for any i_α, then we see that the first column of this matrix is all 0, and hence the determinant is 0. On the other hand, if $j = i_\alpha$, then we see that the first column of the matrix has a 1 in the α-th position, and 0 otherwise. Therefore the determinant $\det(\phi_{i_r}(X_s)) = (-1)^{\alpha-1} * \det M$ where M is obtained from taking out the first column and α-th row in the matrix $(\phi_{i_r}(X_s))$. However, $\det M$ is precisely $\phi_{i_1} \wedge ... \wedge \widehat{\phi_{i_\alpha}} \wedge ... \wedge \phi_{i_k}(X_2, ..., X_k)$, and hence we see that $v_j \lrcorner (\phi_{i_1} \wedge ... \wedge \phi_{i_k}) = (-1)^{\alpha-1} \phi_{i_1} \wedge ... \wedge \widehat{\phi_{i_\alpha}} \wedge ... \wedge \phi_{i_k}$ when $j = i_\alpha$.

(c) We will use part (b) to prove this. Since $\Omega^k(V)$ has basis $\{\phi_{i_1} \wedge ... \wedge \phi_{i_k} : i_1 < ... < i_k\}$ and similarly $\Omega^l(V)$ has basis $\{\phi_{i_1} \wedge ... \wedge \phi_{i_l} : i_1 < ... < i_l\}$, and both sides of the equation

$$v \lrcorner (\omega_1 \wedge \omega_2) = (v \lrcorner \omega_1) \wedge \omega_2 + (-1)^k \omega_1 \wedge (v \lrcorner \omega_2)$$

are linear (in the "v" entry, the "ω_1" entry, and "ω_2" entry), we can just verify it for $v = v_r$ for some r, $\omega_1 = \phi_{i_1} \wedge ... \wedge \phi_{i_k}$, and $\omega_2 = \phi_{j_1} \wedge ... \wedge \phi_{j_l}$. Then part (b) tells us that if $r \neq i_\alpha, j_\alpha$ for any i_α, j_α, then both sides equal 0. On the other hand, if $r = i_\alpha$, then

$$\begin{aligned} v_r \lrcorner (\omega_1 \wedge \omega_2) &= (-1)^{\alpha-1} \phi_{i_1} \wedge ... \wedge \widehat{\phi_{i_\alpha}} \wedge ... \wedge \phi_{i_k} \wedge \phi_{j_1} \wedge ... \wedge \phi_{j_l} \\ &= ((-1)^{\alpha-1} \phi_{i_1} \wedge ... \wedge \widehat{\phi_{i_\alpha}} \wedge ... \wedge \phi_{i_k}) \wedge (\phi_{j_1} \wedge ... \wedge \phi_{j_l}) + 0 \\ &= (v_r \lrcorner \omega_1) \wedge \omega_2 + (-1)^k \omega_1 \wedge (v_r \lrcorner \omega_2) \end{aligned}$$

where the last term in the last equation is 0, while if $r = j_\beta$, then

$$\begin{aligned} v_r \lrcorner (\omega_1 \wedge \omega_2) &= (-1)^{\beta+k-1} \phi_{i_1} \wedge ... \wedge \phi_{i_k} \wedge \phi_{j_1} \wedge ... \wedge \widehat{\phi_{j_\beta}} \wedge ... \wedge \phi_{j_l} \\ &= 0 + (-1)^k (\phi_{i_1} \wedge ... \wedge \phi_{i_k}) \wedge (-1)^{\beta-1} (\phi_{j_1} \wedge ... \wedge \widehat{\phi_{j_\beta}} \wedge ... \wedge \phi_{j_l}) \\ &= (v_r \lrcorner \omega_1) \wedge \omega_2 + (-1)^k \omega_1 \wedge (v_r \lrcorner \omega_2) \end{aligned}$$

where the first term in the last equation is 0. Thus we conclude that

$$v \lrcorner (\omega_1 \wedge \omega_2) = (v \lrcorner \omega_1) \wedge \omega_2 + (-1)^k \omega_1 \wedge (v \lrcorner \omega_2)$$

Question 7-5.

Let $F : M \to \mathbb{R}^n$ be defined as $F = (f^1, ..., f^n)$ where $f^i : M \to \mathbb{R}$ are smooth. Given $p \in M$, let (x, U_0) be a coordinate system of M. Now f^i form a coordinate system in a neighborhood U of $p \in M$ if and only if $F|_U$ is a diffeomorphism from U onto an open subset V of $\mathbb{R}^n$, and this happens if and only if $DF(p)$ is a nonsingular matrix (by Inverse Function Theorem). Now $df_1 \wedge ... \wedge df_n(p) \neq 0$ if and only if $df_1 \wedge ... \wedge df_n(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}) \neq 0$ (this follows from question 7-16, which will be proved later). But

$$df_1 \wedge ... \wedge df_n \left(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n} \right) = \det \left(df^j \left(\frac{\partial}{\partial x^i} \right) \right) = \det \left(\frac{\partial f^j}{\partial x^i} \right)$$

and this last expression is precisely $\det(DF(p))$, and hence we see that $df_1 \wedge ... \wedge df_n(p) \neq 0$ if and only if f^i form a coordinate system in a neighborhood U of $p \in M$.

Question 7-6.

(a) For dim $V = 1$ and 2, $\Omega^2(V)$ is 0-dimensional and 1-dimensional, respectively, and hence every $\omega \in \Omega^2(V)$ is automatically decomposable. So suppose dim $V = 3$. Let $\epsilon^1, \epsilon^2, \epsilon^3$ be a dual basis for V, then every $\omega \in \Omega^2(V)$ can be written as $a\epsilon^1 \wedge \epsilon^2 + b\epsilon^1 \wedge \epsilon^3 + c\epsilon^2 \wedge \epsilon^3$. However,

$$\begin{aligned}
&\left((a+1)\epsilon^1 + \epsilon^2 + \left(\frac{b+c}{a} + c\right)\epsilon^3\right) \wedge \left(\epsilon^1 + \epsilon^2 + \left(\frac{b+c}{a}\right)\epsilon^3\right) \\
&= (a+1)\epsilon^1 \wedge \epsilon^2 + (a+1)\left(\frac{b+c}{a}\right)\epsilon^1 \wedge \epsilon^3 + \epsilon^2 \wedge \epsilon^1 + \left(\frac{b+c}{a}\right)\epsilon^2 \wedge \epsilon^3 + \left(\frac{b+c}{a} + c\right)\epsilon^3 \wedge \epsilon^1 + \left(\frac{b+c}{a} + c\right)\epsilon^3 \wedge \epsilon^2 \\
&= a\epsilon^1 \wedge \epsilon^2 + b\epsilon^1 \wedge \epsilon^3 + c\epsilon^2 \wedge \epsilon^3
\end{aligned}$$

This shows that $a\epsilon^1 \wedge \epsilon^2 + b\epsilon^1 \wedge \epsilon^3 + c\epsilon^2 \wedge \epsilon^3 = \left((a+1)\epsilon^1 + \epsilon^2 + \left(\frac{b+c}{a} + c\right)\epsilon^3\right) \wedge \left(\epsilon^1 + \epsilon^2 + \left(\frac{b+c}{a}\right)\epsilon^3\right)$ is decomposable as the wedge product of two elements in $\Omega^1(V)$, and thus we conclude that for dim $V \le 3$, every $\omega \in \Omega^2(V)$ is decomposable.

(b) Suppose $\phi_i, i = 1, ..., 4$ are independent, and $\omega = (\phi_1 \wedge \phi_2) + (\phi_3 \wedge \phi_4)$. We will show that ω is not decomposable. We compute $\omega \wedge \omega$.

$$\begin{aligned}
\omega \wedge \omega &= ((\phi_1 \wedge \phi_2) + (\phi_3 \wedge \phi_4)) \wedge ((\phi_1 \wedge \phi_2) + (\phi_3 \wedge \phi_4)) \\
&= 2\phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4
\end{aligned}$$

If ω is decomposible, then $\omega \wedge \omega = 0$, so we see that ω cannot be decomposable. This shows that for dim $V \ge 4$ there are elements $\omega \in \Omega^2(V)$ that is NOT decomposable.

Question 7-16.

(a) Since ϵ^I (where I is an increasing multi-index of length k) forms a basis of $\Omega^k(V)$, we can write $\omega = \sum_I c_I \epsilon^I$. Now as $w_i = \sum_{j=1}^n \alpha_{ji} v_j$ (where $i = 1, ..., k$), we let $T = (\alpha_{ji})$ be the matrix consisting of entries α_{ji}, then $w_i = Tv_i$ for all $i = 1, ..., k$. Then

$$\begin{aligned}
\omega(w_1, ..., w_k) &= \omega(Tv_1, ..., Tv_k) \\
&= \sum_I c_I \epsilon^I (Tv_1, ..., Tv_k) \\
&= \sum_I c_I \alpha_I
\end{aligned}$$

where α_I is the determinant of the $k \times k$ minor of T obtained by selecting rows $i_1, ..., i_k$. On the other hand, given $i_1 < ... < i_k$, we have $\omega(v_{i_1}, ..., v_{i_k}) = \sum_I c_I \epsilon^I (v_{i_1}, ..., v_{i_k}) = c_{i_1 < ... < i_k}$. Thus we see that $\omega(w_1, ..., w_k) = \sum_I c_I \alpha_I = \sum_I \alpha_I \omega(v_{i_1}, ..., v_{i_k})$.

(b) If $f : M \to N$ is a C^∞ function between n-manifolds, then $f^*(gdy^{i_1} \wedge ... \wedge dy^{i_k}) = (g \circ f) d(y^{i_1} \circ f) \wedge ... \wedge d(y^{i_k} \circ f)$ (Since f^* is linear, we will check only on basis elements $dy^{i_1} \wedge ... \wedge dy^{i_k}$). By part (a), a k-form is determined by where it sends $(v_{i_1}, ..., v_{i_k})$ for $i_1 < ... < i_k$ an increasing multi-index. Now

$$(g \circ f) d(y^{i_1} \circ f) \wedge ... \wedge d(y^{i_k} \circ f) \left(\frac{\partial}{\partial x^{j_1}}, ..., \frac{\partial}{\partial x^{j_k}}\right) = (g \circ f) \det \left(\frac{\partial (y^{i_\alpha} \circ f)}{\partial x^{j_\beta}}\right)$$

Now since $dx^{j_1} \wedge ... \wedge dx^{j_k} (\frac{\partial}{\partial x^{j_1}}, ..., \frac{\partial}{\partial x^{j_k}}) = 1$, we see that

$$f^*(gdy^{i_1} \wedge ... \wedge dy^{i_k}) = \sum_J (g \circ f) \det \left(\frac{\partial (y^{i_\alpha} \circ f)}{\partial x^{j_\beta}}\right) dx^{j_1} \wedge ... \wedge dx^{j_k}$$

and this is a generalization of Theorem 7. From what we proved in this part, letting $f = Id$, corollary 8 now generalizes to: If (x, U) and (y, V) are two coordinate systems on M then

$$gdy^{i_1} \wedge ... \wedge dy^{i_k} = \sum_J g \cdot \det \left(\frac{\partial (y^{i_\alpha})}{\partial x^{j_\beta}}\right) dx^{j_1} \wedge ... \wedge dx^{j_k}$$

(c)

$$
\begin{aligned}
d(gdy^{i_1}\wedge\ldots\wedge dy^{i_k}) &= dg\wedge dy^{i_1}\wedge\ldots\wedge dy^{i_k} = \sum_i\left(\frac{\partial g}{\partial x^i}dx^i\right)\wedge\sum_J\det\left(\frac{\partial(y^{i_\alpha})}{\partial x^{j_\beta}}\right)dx^{j_1}\wedge\ldots\wedge dx^{j_k}\\
&= \sum_{i\notin J}\sum_J\det\left(\frac{\partial(y^{i_\alpha})}{\partial x^{j_\beta}}\right)\left(\frac{\partial g}{\partial x^i}dx^i\right)\wedge dx^{j_1}\wedge\ldots\wedge dx^{j_k} = \sum_J\sum_i\det\left(\frac{\partial(y^{i_\alpha})}{\partial x^{j_\beta}}\right)\left(\frac{\partial g}{\partial x^i}dx^i\right)\wedge dx^{j_1}\wedge\ldots\wedge dx^{j_k}\\
&= \sum_J dg\det\left(\frac{\partial(y^{i_\alpha})}{\partial x^{j_\beta}}\right)\wedge dx^{j_1}\wedge\ldots\wedge dx^{j_k} = d\left(\sum_J g\cdot\det\left(\frac{\partial(y^{i_\alpha})}{\partial x^{j_\beta}}\right)dx^{j_1}\wedge\ldots\wedge dx^{j_k}\right)
\end{aligned}
$$

So we see that d is coordinate-independent.

Question 7-27.

(a) If $\nabla f(p) = w_p$, then

$$
\begin{aligned}
(D_v f)(p) &= \left.\frac{d}{dt}\right|_{t=0} f(p+tv)\\
&= \sum v^i\frac{\partial f}{\partial x^i}\\
&= \left\langle\sum v^i\frac{\partial}{\partial x^i},\sum\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^i}\right\rangle\\
&= \langle v,w\rangle
\end{aligned}
$$

Because $\langle v,w\rangle$ attains maximum value when $v = cw$ for some c, we see that $\nabla f(p)$ is the direction in which f is changing the fastest at p.

(b)

$$
df = \frac{\partial f}{\partial x^1}dx^1+\frac{\partial f}{\partial x^2}dx^2+\frac{\partial f}{\partial x^3}dx^3 = w_{\nabla f}
$$

$$
\begin{aligned}
d(\omega_X) &= d(a^1dx^1+a^2dx^2+a^3dx^3)\\
&= da^1\wedge dx^1+da^2\wedge dx^2+da^3\wedge dx^3\\
&= \sum_i\sum_j\frac{\partial a^i}{\partial x^j}dx^j\wedge dx^i\\
&= \left(\frac{\partial a^2}{\partial x^1}-\frac{\partial a^1}{\partial x^2}\right)dx^1\wedge dx^2+\left(\frac{\partial a^3}{\partial x^1}-\frac{\partial a^1}{\partial x^3}\right)dx^1\wedge dx^3+\left(\frac{\partial a^3}{\partial x^2}-\frac{\partial a^2}{\partial x^3}\right)dx^2\wedge dx^3\\
&= \eta_{\text{curl } X}
\end{aligned}
$$

$$
\begin{aligned}
d(\eta_X) &= d(a^1dx^2\wedge dx^3-a^2dx^1\wedge dx^3+a^3dx^1\wedge dx^2)\\
&= da^1\wedge dx^2\wedge dx^3-da^2\wedge dx^1\wedge dx^3+da^3\wedge dx^1\wedge dx^2\\
&= \frac{\partial a^1}{\partial x^1}dx^1\wedge dx^2\wedge dx^3-\frac{\partial a^2}{\partial x^2}dx^2\wedge dx^1\wedge dx^3+\frac{\partial a^3}{\partial x^3}dx^3\wedge dx^1\wedge dx^2\\
&= \left(\frac{\partial a^1}{\partial x^1}+\frac{\partial a^2}{\partial x^2}+\frac{\partial a^3}{\partial x^3}\right)dx^1\wedge dx^2\wedge dx^3\\
&= (\text{div } X)dx^1\wedge dx^2\wedge dx^3
\end{aligned}
$$

(c) Now as $d \circ d = 0$, first of all $d \circ df = d(w_{\nabla f}) = \eta_{\text{curl } \nabla f} = 0$, so we see that curl $\nabla f = 0$. On the other hand, $d \circ d(w_X) = d(\eta_{\text{curl } X}) = (\text{div curl } X) dx \wedge dy \wedge dz = 0$, we see that div curl $X = 0$.

(d) It was shown in Spivak that if M is smoothly contractible to a point, then every closed form on M is exact. Now as star shaped open set U is smoothly contractible to a point, we see that forms of U are exact. Therefore, if curl $X = 0$, then $\eta_{\text{curl } X} = d(w_X) = 0$, so $w_X = df$ for some f, but $df = w_{\nabla f}$, so $X = \nabla f$ for some f. Similarly, if div $X = 0$, then $d(\eta_X) = (\text{div } X) dx^1 \wedge dx^2 \wedge dx^3 = 0$, so we see that $\eta_X = d(w_Y)$ for some w_Y. But $d(w_Y) = \eta_{\text{curl } Y}$, so we see that $X = \text{curl } Y$ for some Y on U.