Math 214 Problem Set 4

Kuan-Ying Fang

October 31, 2011

Question 5-4.

Suppose $f : (-c, c) \times U \times V \to \mathbb{R}^n$ is C^∞, where $U, V \subset \mathbb{R}^n$ are open, and let $(x_0, y_0) \in U \times V$. We define another function $F : (-c, c) \times U \times V \to \mathbb{R}^n \times \mathbb{R}^n$ by taking $F(t, x, y) = (y, f(t, x, y))$. Let π_1, π_2 be projection of $\mathbb{R}^n \times \mathbb{R}^n$ to the first and second factor respectively, then since $\pi_2 \circ F = f$ is C^∞ and $\pi_1 \circ F(t, x, y) = y$ is also C^∞, we see that F is C^∞. We apply F to Theorem 2 (the existence of solutions to ODE) in Spivak, we see that since F is C^∞, then it follows that there exist a neighborhood W of (x_0, y_0) such that F is bounded and Lipschitz, and hence by Theorem 2 (uniqueness by Theorem 3) there exist a number $b > 0$ such that for each $(x, y) \in W$ there is a unique $\gamma_{(x,y)} : (-b, b) \to U \times V$ with $\gamma'_{(x,y)}(t) = F(t, \gamma_{(x,y)}(t))$ and $\gamma_{(x,y)} = (x, y)$. However, if we denote $\gamma_{(x,y)}(t) = (\alpha_{(x,y)}(t), \beta_{(x,y)}(t))$, then $\gamma'_{(x,y)}(t) = F(t, \gamma_{(x,y)}(t))$ and $\gamma_{(x,y)} = (x, y)$ translates to a system of equations $\alpha'_{(x,y)}(t) = \beta_{(x,y)}(t)$, $\beta'_{(x,y)}(t) = f(t, \alpha_{(x,y)}(t), \beta_{(x,y)}(t))$, and the condition $\alpha_{(x,y)}(0) = x$ and $\beta_{(x,y)}(0) = y$. But these, again, translates to $\alpha''_{(x,y)}(t) = f(t, \alpha_{(x,y)}(t), \alpha'_{(x,y)}(t))$, $\alpha_{(x,y)}(0) = x$, and $\alpha'_{(x,y)}(0) = y$. Which is what we wanted to prove.

Question 5-9.

We will show that $\lim_{h\to 0}(\phi_h)_* Y_{\phi_{-h}(p)} = Y_p$. Let (U, x) be a coordinate chart around p, and denote $x = (x^1, ..., x^n)$. Now in coordintes, we see that $(\phi_h)_*$ at $\phi_{-h}(p)$ is the matrix

$$(\phi_h)_* = \begin{pmatrix} \frac{\partial \phi_h^1}{\partial x^1}(\phi_{-h}(p)) & \cdots & \frac{\partial \phi_h^1}{\partial x^n}(\phi_{-h}(p)) \\ & \vdots & \\ \frac{\partial \phi_h^n}{\partial x^1}(\phi_{-h}(p)) & \cdots & \frac{\partial \phi_h^n}{\partial x^n}(\phi_{-h}(p)) \end{pmatrix}$$

and hence it sends $Y_{\phi_{-h}(p)} = (Y^1_{\phi_{-h}(p)}, ..., Y^n_{\phi_{-h}(p)})$ to

$$\sum_{i=1}^n \left(\sum_{j=1}^n \frac{\partial \phi_h^i}{\partial x^j}(\phi_{-h}(p)) Y^j_{\phi_{-h}(p)} \frac{\partial}{\partial x^i} \right)$$

Now note $lim_{h\to 0} Y^j_{\phi_{-h}(p)} = Y^j_p$ (as Y is a C^∞ vector field) and $\lim_{h\to 0} \frac{\partial \phi_h^i}{\partial x^j}(\phi_{-h}(p)) = \frac{\partial \lim_{h\to 0} \phi_h^i(\phi_{-h}(p))}{\partial x^j} = \delta^i_j$ where δ^i_j is the Kronecker delta (because $\lim_{h\to 0} \phi_h(\phi_{-h}(p))$ is the identity function). Thus we see that the sum above simplifies to

$$\sum_{i=i}^n Y^i_p \frac{\partial}{\partial x^i} = Y_p$$

Question 5-10.

(a) We will show that $L_X(f\cdot\omega) = Xf\cdot\omega + f\cdot L_X\omega$.

$$\begin{aligned}
L_X(f\cdot\omega)_p(X_p) &= \lim_{h\to 0}\frac{(\phi_h^* f\cdot\omega)_p(X_p) - (f\cdot\omega)_p(X_p)}{h}\\
&= \lim_{h\to 0}\frac{f(\phi_h(p))(\phi_h^*\omega)_p(X_p) - f(p)(\omega)_p(X_p)}{h}\\
&= \lim_{h\to 0}\frac{f(\phi_h(p))(\phi_h^*\omega)_p(X_p) - f(p)(\phi_h^*\omega)_p(X_p) + f(p)(\phi_h^*\omega)_p(X_p) - f(p)(\omega)_p(X_p)}{h}\\
&= \lim_{h\to 0}\frac{f(\phi_h(p))(\phi_h^*\omega)_p(X_p) - f(p)(\phi_h^*\omega)_p(X_p)}{h} + \lim_{h\to}\frac{f(p)(\phi_h^*\omega)_p(X_p) - f(p)(\omega)_p(X_p)}{h}\\
&= \lim_{h\to 0}\frac{f(\phi_h(p)) - f(p)}{h}(\phi_h^*\omega)_p(X_p) + f(p)\lim_{h\to}\frac{(\phi_h^*\omega)_p(X_p) - (\omega)_p(X_p)}{h}\\
&= Xf(p)\omega_p(X_p) + f(p)(L_X\omega)_p(X_p)
\end{aligned}$$

for any X_p in the tangent space (sorry bad notation, here X_p and X are not relavant). Thus we see that

$$L_X(f\cdot\omega) = Xf\cdot\omega + f\cdot L_X\omega$$

(b) We will show that $L_X[\omega(Y)] = (L_X\omega)(Y) + \omega(L_X Y)$.

$$\begin{aligned}
&L_X[\omega(Y)] = (L_X\omega)(Y) + \omega(L_XY) =\\
&= \lim_{h\to 0}\frac{\omega(Y)\circ\phi_h(p) - \omega(Y)(p)}{h}\\
&= \lim_{h\to 0}\frac{\omega_{\phi_h(p)}(Y_{\phi_h(p)}) - \omega_p(Y_p)}{h}\\
&= \lim_{h\to 0}\frac{\omega_{\phi_h(p)}(Y_{\phi_h(p)}) - \omega_{\phi_h(p)}(\phi_{h*}Y_p) + \omega_{\phi_h(p)}(\phi_{h*}Y_p) - \omega_p(Y_p)}{h}\\
&= \lim_{h\to 0}\frac{\omega_{\phi_h(p)}(Y_{\phi_h(p)}) - \omega_{\phi_h(p)}(\phi_{h*}Y_p)}{h} + \lim_{h\to 0}\frac{\omega_{\phi_h(p)}(\phi_{h*}Y_p) - \omega_p(Y_p)}{h}\\
&= \lim_{h\to 0}\frac{\omega_{\phi_h(p)}(Y_{\phi_h(p)} - \phi_{h*}Y_{\phi_{-h}\phi_h(p)})}{h} + (L_X\omega)(Y)(p) = \omega(L_XY)(p) + (L_X\omega)(Y)(p)
\end{aligned}$$

So we see that $L_X[\omega(Y)] = (L_X\omega)(Y) + \omega(L_XY)$.

(c) If we change the definition, then we see that part (a) and part (b) becomes $L_X(f\cdot\omega) = -Xf\cdot\omega + f\cdot L_X\omega$ and $L_X[\omega(Y)] = (L_X\omega)(Y) - \omega(L_XY)$

Question 5-12.

The Jacobi identity states

$$[Z,[X,Y]] + [X,[Y,Z]] + [Y,[X,Z]] = 0$$

In the book it is established that $[X,Y] = XY = YX$. Thus given any $f \in C^\infty(M)$, we see that

$$\begin{aligned}
&[Z,[X,Y]] + [X,[Y,Z]] + [Y,[Z,X]](f)\\
&= [Z,[X,Y]]f + [X,[Y,Z]]f + [Y,[Z,X]]f\\
&= [Z, XY - YX]f + [X, YZ - ZY]f + [Y, ZX - XZ]f\\
&= (Z(XY - YX) - (XY - YX)Z)f + (X(YZ - ZY) - (YZ - ZY)X)f + (Y(ZX - XZ) - (ZX - XZ)Y)f\\
&= ZXYf - ZYXf - XYZf + YXZf + XYZf - XZYf - YZXf + ZYXf + YZXf - YXZf - ZXYf + XZYf\\
&= 0
\end{aligned}$$

Since this holds for every $f \in C^\infty(M)$, we see that $[Z,[X,Y]] + [X,[Y,Z]] + [Y,[X,Z]] = 0$.

Question 5-15.

(a) Since we have defined D for functions and vector fields, we will construct it for covariant vector fields, and then for tensor fields of type (k,l). Given a covariant vector field ω, we will use property (3) to define where D takes it. Let $C(\omega \otimes X) = \omega(X)$ be a contraction, then we see by property (2) and (3) that $DC(\omega \otimes X) = CD(\omega \otimes X) = C(D\omega \otimes X + \omega \otimes DX)$, so $D(\omega(X)) = D\omega(X) + \omega(DX)$, and hence we define $D\omega(X) = D(\omega(X)) - \omega(DX)$. Now we extend this to tensor fields of type (k,l). Since all tensor fields of type (k,l) is a of the form $\omega_{i_1} \otimes ... \otimes \omega_{i_k} \otimes X^{j_1} \otimes ... \otimes X^{j_l}$, we extend D by property (2) $\mathbb{R}$-linearly to all tensor fields of type (k,l). Since an operator is determined by where it sends every element, our construction will be the unique construction satisfying (1), (2), and (3). Thus we have a unique extension of D to an operator taking tensor fields of type (k,l) to themselves satisfying (1), (2), and (3).

(b) We will check that $D_A(fY) = f \cdot D_A(Y) + D_A(f) \cdot Y$. Now $D_A(fY) = A(fY)$, so $(A(fY))_p = A_p(f(p)Y_p) = f(p)A_p(Y_p)$, so $D_A(fY) = f \cdot A(Y) = 0 + f \cdot D_A(Y) = D_A(f) \cdot Y + f \cdot D_A(Y)$. Now using part (a), we see that there is a unique extension of D_A satisfying (1), (2), (3).

(c) $(D_A\omega)_p(X_p) = D_A(\omega(X))(p) - \omega_p((D_AX)_p) = 0 - \omega_p((AX)_p) = -\omega_p(A_p(X_p)) = -(A_p^*\omega_p)(X_p)$. So we see that $(D_A\omega)_p = -(A_p^*\omega_p)$.

(d) $L_{fX}V = [fX,V] = fXV - V(fX) = fXV - VfX - fXV = f[X,V] - Vf \cdot X = fL_XV - Vf \cdot X$. On the other hand, we see that $(D_{X\otimes df}V)_p = (X \otimes df)_p(V_p)$, we will figure out what $(X \otimes df)_p(V_p)$ is by seeing how the covectors act on it. Given any λ_p, we see that $\lambda_p((X \otimes df)_p(V_p)) = \lambda_p(X_p)df_p(V_p)$ by definition, and hence in coordinates we let $dx^1, ..., dx^n$ be the local nowhere-dependent section associate to the coordinates $x^1, ..., x^n$, then we see that $dx^i((X \otimes df)_p(V_p)) = dx^i(X_p)df_p(V_p) = X_p^iV_p(f)$ for all $i = 1, ..., n$, so we see that $(X \otimes df)_p(V_p) = V_p(f)\sum X_p^i \frac{\partial}{\partial x^i}|_p = V_p(f) \cdot X_p$, and so we see that $D_{X\otimes df}V = Vf \cdot X$. Thus we have $L_{fX}V = fL_XV - Vf \cdot X$.

(e) If T is of type $(2,1)$, then we see that T can be written in coordinates as

$$T = \sum_{k,l,m} T_k^{lm} dx^k \otimes \frac{\partial}{\partial x^l} \otimes \frac{\partial}{\partial x^m}$$

and we write

$$A = \sum_{i,j} A_i^j dx^i \otimes \frac{\partial}{\partial x^j}$$

Then we will use property (2) to deduce that

$$\begin{aligned}(D_AT) = &\sum_{i,j} A_i^j \sum_{k,l,m} T_k^{lm} dx^k \otimes dx^i \left(\frac{\partial}{\partial x^l}\right) \otimes \frac{\partial}{\partial x^m} \otimes \frac{\partial}{\partial x^j} \\ &+ \sum_{i,j} A_i^j \sum_{k,l,m} T_k^{lm} dx^k \otimes \frac{\partial}{\partial x^l} \otimes dx^i \left(\frac{\partial}{\partial x^m}\right) \otimes \frac{\partial}{\partial x^j} \\ &- \sum_{i,j} A_i^j \sum_{k,l,m} T_k^{lm} dx^k \otimes \frac{\partial}{\partial x^l} \otimes \frac{\partial}{\partial x^m} \otimes dx^k \left(\frac{\partial}{\partial x^j}\right)\end{aligned}$$

So we see that

$$(D_AT)_\alpha^{\beta\gamma} = \sum_l A_l^\beta T_\alpha^{l\gamma} + \sum_m A_m^\gamma T_\alpha^{\beta m} - \sum_j A_\alpha^j T_j^{\beta\gamma}$$

Question 5-19.

(a) Suppose M is compact, then $f^{-1}(0)$ is a closed subset of a compact set, hence compact. Since 0 is a regular value of $f : M \to \mathbb{R}$, we let $p \in f^{-1}(0)$, and using Theorem 2.9 we let (x_p, U_p) be a coordinate system around p such that $f \circ x_p^{-1}(a^1, ..., a^n) = (a^1)$. We do this for all $p \in f^{-1}(0)$, and we obtain $\{U_p\}_{p \in f^{-1}(0)}$ with charts $\{x_p\}_{p \in f^{-1}(0)}$. Since $f^{-1}(0) \subset \cup_{p \in f^{-1}(0)} U_p$, by compactness there is a finite subcover $\{U_{p_i}\}_{i=1,...,n}$, which we denote the union by V. Now we let ψ_{p_i} be a partition of unity subordinate to this finite cover, and define a vector field at $q \in V$ by $\sum_{q \in U_{p_i}} \psi_{p_i}(q)(x^{-1})_*(\frac{\partial}{\partial (x_{p_i})_1}|_{x_{p_i}(q)})$. Note that for each x_{p_i}, we have $f_*(x_{p_i}^{-1})_*(\frac{\partial}{\partial (x_{p_i})_1}|_{x_{p_i}(q)}) = (f \circ x_{p_i}^{-1})_*(\frac{\partial}{\partial (x_{p_i})_1}|_{x_{p_i}(q)}) = \frac{d}{dt}|_{f(q)}$ as $f \circ x_{p_i}^{-1}$ projects the first component. Thus we see that we have defined a vector field X on a neighborhood V of $f^{-1}(0)$ such that $f_* X = \frac{d}{dt}$. Now we let ϕ be the flow of X and θ be the flow of $\frac{d}{dt}$. Now as $f_* X = \frac{d}{dt}$, we see that for some t such that ϕ_t and θ_t exists, we have $\theta_t \circ f(p) = f \circ \phi_t(p)$ which is equivalent to $\theta(f(p), t) = f(\phi(p, t))$ (here $\theta(f(p), t) = f(p) + t$ because the flow is $\frac{d}{dt}$). So in particular, we see that around $p \in f^{-1}(0)$, the condition $\theta(f(p), t) = f(\phi(p, t))$ tells us that we have a diffeomorphism $\phi : (-t, t) \times f^{-1}(0) \to f^{-1}((-t, t))$ such that $f(\phi(p, t)) = \theta(f(p), t) = \theta(0, t) = 0 + t = t$.

(b) In general, if M is compact and $q \in N$ is a regular value of $f : M \to N$, then following a very similar argument, we can prove that there is a neighborhood U of q and a diffeomorphism $\phi : f^{-1}(q) \times U \to f^{-1}(U)$ with $f(\phi(p, q')) = q'$. Again we can pick coordinates (x, V) around $p \in f^{-1}(q)$ and (y, W) around q such that $y \circ f \circ x^{-1}(a^1, ..., a^n) = (a^1, ..., a^m)$. Then we can define m commuting vector fields $x_*^{-1}(\frac{\partial}{\partial x_1}), ..., x_*^{-1}(\frac{\partial}{\partial x_m})$ on each V, and using partition of unity to glue each of them, we get m commuting independent vector fields $X^1, ..., X^k$ such that $(yf)_* X^i = \frac{\partial}{\partial x_i}$. Then in coordinates, we define θ_i to be the flow of $(y^{-1})_* \frac{\partial}{\partial x_i}$ and ϕ_i to be the flow of X^i, we define $\Psi = (\phi_1)_{t_1} \circ (\phi_2)_{t_2} \circ ... \circ (\phi_m)_{t_m}$ where ϕ_i is defined on t_i (here the orders do not matter as these vector fields commute), then similarly to part (a), we see that $\Psi : f^{-1}(q) \times U \to f^{-1}(U)$ is a diffeomorphism (where $U \subset y^{-1}((-t_1, t_1) \times ... \times (-t_m, t_m)))$ and if $q' = y^{-1}(h_1, ..., h_m) \in U$ then $f \circ \Psi(p, q') = f \circ (\phi_1)_{h_1} \circ (\phi_2)_{h_2} \circ ... \circ (\phi_m)_{h_m}(p) = (\theta_1)_{h_1} \circ (\theta_2)_{h_2} \circ ... \circ (\theta_m)_{h_m}(f(p)) = y^{-1}(h_1, ..., h_m) = q'$.

Question 6-1.

(a) We will show that a k-dimensional distribution on M is just a subbundle of TM. By definition, a k-dimensional distribution on M is a function $\Psi : p \mapsto \Delta_p$ where Δ_p is a k-dimensional subspace of T_pM, and for any neighborhood U and k vector fields $X_1, ...X_k$ such that $X_1(q), ..., X_k(q)$ are a basis for Δ_q for all $q \in U$. First of all, this defines a k-plane bundle with local trivializations induced from $X_1, ..., X_k$ (because $X_1, ..., X_k$ are nowhere dependent local sections of the tangent bundle, and we use question 3.7 of HW2). Now as the diagram

$$\begin{tikzcd} \Delta \arrow[r, "i"] \arrow[d] & T_pM \arrow[d] \\ M \arrow[r, "1_M"] & M \end{tikzcd}$$

commutes, and linearity on the fiber holds because the map i is an inclusion, we see that a k-dimensional distribution on M is actually a subbundle.

(b) We define a smooth subbundle to be a subbundle such that the $(i, 1_B)$ is a smooth map. Suppose a k-dimensional distribution on M is C^∞, then let $X_1, ..., X_k$ be a smooth nowhere dependent local sections of Δ defined on a neighborhood U of p, then we can extend these into a smooth local frame, and again by question 3.7 of HW2 we see that there is a local trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ defined by $\Phi(\sum_i a^i (X_i)_q) = (q, (a^1, ..., a^k))$, which is smooth because all the X_i are. On the other hand, let $\Phi' : U \times \mathbb{R}^n \to \pi'^{-1}(U)$ be the smooth local trivialization of TM, the tangent bundle, then we see that $\Phi' \circ \Phi$ is a smooth map, which, on $\pi^{-1}(U)$ is the inclusion map. Thus a smooth k-dimensional distribution on M is a smooth subbundle. On the other hand, if the inclusion is a smooth map, then the k-dimensional distribution on M is a smooth distribution because for any p we can just take the smooth local trivializations on U containing p, and then 3.7(c) shows that these are smooth nowhere dependent sections.

Question 6-2.

(a) Referring back to Theorem 5.2, we let $x_0 = \alpha(0, 0, ..., 0, 0)$, then as f_2 is C^∞, it is Lipschitz, so for some $a > 0$, f_2 automatically satisfies condition (1) and (2) in Theorem 5.2 on $\bar{B}_{2a}(x_0)$. Then Theorem 5.2 states that for all $x \in \overline{B}_a(x_0)$ there is a unique $\beta_2 : (-b, b) \to V$ such that $\beta_2(0) = x$ and $\beta_2^{'}(t) = f_2(t_1, t, 0, ..., 0, \beta_2(t))$. We can pick ϵ_1 small enough such that $\alpha(t^1, 0, ..., 0) \in \bar{B}_{a/2}(x_0)$ for all $|t_1| < \epsilon_1$ (we can do this by continuity of $\alpha(t, 0, 0, ..., 0) = \beta_1(t)$). Then for $|t_1| < \epsilon_1$ by theorem 5.2 we let $\epsilon_2 = b$ defined above, then $\beta_2(t)$ with initial condition $\beta_2(0) = \alpha(t^1, 0, ..., 0)$ is defined for all $|t| < \epsilon_2$. This justifies the "pick ϵ_1 small enough" statement.

(b) Since in the proof we construct the solution α by solving n first order differential equation in n-steps (each step we solve one first order differential equation, the solution of which is used in the next step), by Theorem 5.3, which proves the uniqueness of solution for first order differential equations, we see that α is unique because in each step the solution is.