Math 214 Problem Set 3

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Question 4-2.

Suppose $f, g : M \to \mathbb{R}$ are C^∞, then given any point $p \in M$ and $X_p \in T_pM$,

$$\begin{aligned} d(fg)_p(X_p) &= X_p(fg) = (X_pf)g + f(X_pg) \\ &= df_p(X_p)g + fdg_p(X_p) \end{aligned}$$

where the second equality follows because X_p is a derivation. So we see that $d(fg)_p = g\ df_p + f\ dg_p$ for all p, and hence $d(fg) = g\ df + f\ dg$

Question 4-4.

(a) Suppose that $v_1, ..., v_n$ and $w_1, ..., w_n$ are basis for V which are equally oriented, i.e. the transition matrix (B_i^j) such that $v_i = \sum_j B_i^j w_j$ has positive determinant. Then consider the dual basis $v_1^*, ..., v_n^*$ and $w_1^*, ..., w_n^*$ for V^* corresponding to $v_1, ..., v_n$ and $w_1, ..., w_n$. We see that $w_k^*(v_i) = w_k^*(\sum_j B_i^j w_j) = \sum_j B_i^j w_k^*(w_j) = B_i^k$. Thus we see that $w_k^* = \sum_k B_i^k v_i^*$. Therefore the transition matrix for the dual basis $v_1^*, ..., v_n^*$ and $w_1^*, ..., w_n^*$ are the transpose of the original matrix $(B_i^j)^T$. Now the determinant of the transpose of a matrix is equal to the determinant of the original matrix. Thus we see that the dual basis $v_1^*, ..., v_n^*$ and $w_1^*, ..., w_n^*$ are equally oriented.

(b) The previous part of this question shows that if a linear isomorphism $g : (V, \mu) \to (W, \eta)$ is order preserving, then letting $v_1, v_2, ..., v_n$ be a basis of V such that $[v_1, ..., v_n] = \mu$ (then $f(v_1), ..., f(v_n)$ is a basis for W and we know $[f(v_1), ..., f(v_n)] = \eta$), we see that $g^* : (W^*, \eta') \to (V^*, \mu')$ where $[f(v_1)^*, ..., f(v_n)^*] = \eta'$ and $[v_1^*, ..., v_n^*] = \mu'$ is order perserving. Suppose $\xi : E \to M$ is an orientable bundle, and let μ be an orientation of the top space E. Then given any trivialization ϕ of ξ on an open subset $U \subset M$ such that ϕ is order preserving on each fibre, we have an induced trivialization $\psi = (\phi^*)^{-1}$ for ξ^* on U which is also order preserving on each fibre (given we pick the orientation η on E^* such that the dual of a basis in μ is in η). Thus we conclude that ξ is orientable if and only if ξ^* is orientable.

Question 4-6.

(a) We will show that under the natural isomorphism $V \to V^*$ by sending $i_V(v)(\lambda) = \lambda(v)$, the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "i_V"] \arrow[d, "f"'] & V^{**} \arrow[d, "f^{**}"'] \\ W \arrow[r, "i_W"] & W^{**} \end{tikzcd}$$

Now suppose $v \in V$, then following the lower-left arrows we see that $i_W \circ f(v) \in W^{**}$ takes any $\lambda' \in W^*$ to $i_W \circ f(v)(\lambda') = \lambda'(f(v))$.

On the other hand, following the top-right arrows we see that $f^{**} \circ i_V(v) \in W^{**}$ takes any $\lambda \in W^*$ to $f^{**} \circ i_V(v)(\lambda) = i_V(v)(f^*\lambda)$ by definition of f^{**} being a dual to f^*. But $i_V(v)(f^*\lambda) = (f^*\lambda)(v) = \lambda(f(v))$. Thus we see that, when comapring to the previous paragraph, $i_W \circ f(v) = f^{**} \circ i_V(v)$ and so the diagram commutes.

(b) We will show that there does not exist isomorphisms $i_V : V \to V^*$ such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "i_V"] \arrow[d, "f"'] & V^* \\ W \arrow[r, "i_W"] & W^* \arrow[u, "f^*"'] \end{tikzcd}$$

We will show that this does not even work for $V = \mathbb{R}$. Let $e \in \mathbb{R}$ be the standard basis for $\mathbb{R}$, and let e^* be the dual basis. Suppose $i_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}^*$ takes $e \mapsto ce^*$. Then if the diagram above is to commute for all $f : \mathbb{R} \to \mathbb{R}$, we let $f(e) = c'e$, and follow the diagram. If we go through the top arrow, then we see that $e \mapsto ce^*$, while if we go through the three arrows (the side arrows and the bottom arrow), then we found that $e \mapsto c'cf^*(e^*)$. However, when evaluated at e, we see that $ce^*(e) = c$ and $c'cf^*(e^*)(e) = c'ce^*(f(e)) = c'ce^*(c'e) = c'cc'$. Thus we see that c has to satisfy $c'cc' = c$ for all $c' \in \mathbb{R}$. In $\mathbb{R}$, no such c can possibly exist, so we conclude that there are NO isomorphism $i_V : V \to V^*$ making the above diagram commute.

Question 4-7.

(a) The identity functor satisfies $F(1_V) = 1_V$ and $F(g \circ f) = g \circ f = F(g) \circ F(f)$ essentailly by definition.

(b) Let 1_V be the identity function on V, then $F(1_V) = 1_V^{**}$. We will use part (e) to do this problem. Since $V^{**} = (V^*)^*$, we see by part (e) that $1_V^{**} = (1_V^*)^* = (1_{V^*})^* = 1_{V^{**}}$, so i_V^{**} is the identity function on V^{**}. Now let $f : W \to Z$ and $g : V \to W$, then consider $F(f \circ g) = (f \circ g)^{**}$. Again by part (e), since $(f \circ g)^{**} = ((f \circ g)^*)^* = (g^* \circ f^*)^* = f^{**} \circ g^{**}$. Thus we see that $F(f \circ g) = F(f) \circ F(g)$, and so F is a covariant functor.

(c) By definition $F(1_V)(T)(\lambda_1, ..., \lambda_n) = T(\lambda_1 \circ 1_V, ..., \lambda_n \circ 1_V) = T(\lambda_1, ..., \lambda_n)$, and so we see that $F(1_V) = 1_{\mathfrak{T}_k(V)}$. On the other hand, $F(f \circ g)(T)(\lambda_1, ..., \lambda_n) = (T)(\lambda_1 \circ f \circ g, ..., \lambda_n \circ f \circ g) = F(g)(T)(\lambda_1 \circ f, ..., \lambda_n \circ f) = (F(f) \circ F(g))(T)(\lambda_1, ..., \lambda_n)$, and hence $F(f \circ g) = F(f) \circ F(g)$. So F is a covariant functor.

(d) Suppose F is a functor and $f : V \to W$ is an isomorphism. We will show that $F(f) : F(V) \to F(W)$ (suppose F is covariant) is an isomorphism. The proof of the case when F contravariant is similar. Note that $F(f) \circ F(f^{-1}) = F(f \circ f^{-1}) = 1_{F(W)}$ and $F(f^{-1}) \circ F(f) = F(f^{-1} \circ f) = 1_{F(V)}$ and so we see that $F(f)$ is a morphism in the target category from $F(V) \to F(W)$ with inverse $F(f^{-1})$, and hence $F(f)$ is an isomorphism in the target category.

(e) Let 1_V be the identity function on V, then $F(1_V) = 1_V^*$. Then $i_V^* w(v) = w(i_V v) = w(v)$ for all $w \in V^*$ and $v \in V$, so $i_V^* w = w$ for all $w \in V^*$, and hence $i_V^* = i_{V^*}$. Now let $f : W \to Z$ and $g : V \to W$, then consider $F(f \circ g) = (f \circ g)^*$. Suppose $v' \in V^*$ and $z \in Z$, we see that $(f \circ g)^*(v')(z) = v'(f \circ g(z)) = f^*v'(g(z)) = g^* \circ f^*v'(z)$, so $F(f \circ g) = (f \circ g)^* = g^* \circ f^* = F(g) \circ F(f)$. So F is a contravariant functor.

(f) The verification follows essentially by part (e). $F(1_V) = 1_V^* : T^k(V) \to T^k(V)$ by sending $F(1_V)T(v_1, ..., v_k) = T(1_V(v_1), ..., 1_V(v_k)) = T(v_1, ..., v_k)$, so it maps $T \mapsto T$ and hence $F(1_V)$ is the identity on $T^k(V)$. On the other hand, $F(f \circ g)T(v_1, ..., v_k) = T((f \circ g)(v_1), ..., (f \circ g)(v_n)) = F(f)T(g(v_1), ..., g(v_n)) = F(g) \circ F(f)T(v_1, ..., v_n)$. Thus $F(f \circ g) = F(g) \circ F(f)$, and hence F is a contravariant functor.

Question 5-3.

We will let the complete metric space be just $\mathbb{R}$ with the regular metric, and we will construct a function such that $d(f(x), f(y)) < d(x, y)$ for all $x, y \in \mathbb{R}$, but f has no fixed point. Let $\phi(t) = e^{-\frac{1}{t^2}}$ for $t \neq 0$ and

$\phi(0) = 0$. Consider the function

$$f(x) = \int_0^x \phi(t)dt + \sqrt{\pi}$$

Since $\phi(t)$ is continuous (thus integrable), we see that $f(x)$ is well defined and continuous. Now since $\phi(t) < 1$ for all $t \in \mathbb{R}$, let $x, y \in \mathbb{R}$ and assume without loss of generality that $x > y$, then

$$|f(x) - f(y)| = \left| \int_y^x \phi(t)dt \right| \leq \int_y^x |\phi(t)|dt < \int_y^x 1dt = |x - y|$$

However, as

$$\int_0^\infty \left(1 - e^{-\frac{1}{t^2}}\right) dt = \sqrt{\pi}$$

we see that

$$\int_0^x \left(1 - e^{-\frac{1}{t^2}}\right) dt < \sqrt{\pi} \text{ for all } x \geq 0$$

and so

$$x < \int_0^x e^{-\frac{1}{t^2}} dt + \sqrt{\pi} = f(x) \text{ for all } x \geq 0$$

Similarly we can verify that $f(x) > x$ for all $x < 0$. Thus $f(x) > x$ for all $x \in \mathbb{R}$ and so f has no fixed point. Thus in the complete metric space $\mathbb{R}$ with the regular metric, there exist a function $f(x)$ such that $d(f(x), f(y)) < d(x, y)$ for all $x, y \in \mathbb{R}$, but f has no fixed point.

Question 1.

Suppose $\omega \in T^*M$ and (x, U) is a chart on M, then in local coordinates,

$$w(p) = \sum_{i=1}^n \omega_i(p) dx^i|_p$$

Now given any integer $k \geq 0$, we will show that ω is C^k if and only if w_i are C^k for all i. Let (x, U) be a chart on U and let $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ be the function that take $\xi_i dx^i|_p \mapsto (p, (\xi_1, ..., \xi_n))$, then we see from our contruction of the smooth structure on T^*M that $\Psi = (x \times Id) \circ \Phi : \pi^{-1}(U) \to \mathbb{R}^{2n}$ by mapping $\xi_i dx^i|_p \mapsto (x^1(p), ..., x^n(p), \xi_1, ..., \xi_n)$ is a chart on T^*M. So in the two charts $(x, U), (\Psi, \pi^{-1}(U))$, we see that the coordinate representation of ω, $\Psi \circ \omega \circ x^{-1}$, maps $(x^1(p), ..., x^n(p)) \mapsto (x^1(p), ..., x^n(p), \omega_1(p), ..., \omega_n(p))$. So in the coordinate representation, it is clear that ω is smooth if and only if ω_i are smooth for all i.

Question 2.

We will show that the $\sim$ defined as in the question is an equivalence relation. First of all, $(p, \alpha, v) \sim (p, \alpha, v)$ because $p = p$ and $v = (Id) \cdot v = g_{\alpha\alpha}(p)v$. Secondly, if $(p, \alpha, v) \sim (q, \beta, w)$, then $p = q$ and $w = g_{\beta\alpha}(p)v$, but the $q = p$ and $g_{\alpha\beta}(p)w = g_{\alpha\beta}(p) \circ g_{\beta\alpha}(p)v = (Id) \cdot v = v$, so $(q, \beta, w) \sim (p, \alpha, v)$. Lastly, suppose $(p, \alpha, v) \sim (q, \beta, w)$ and $(q, \beta, w) \sim (r, \gamma, x)$, then we see that $p = q = r$, $w = g_{\beta\alpha}(p)v$, and $x = g_{\gamma\beta}(p)w$. Now by property (c), we see that $x = g_{\gamma\beta}(p) \circ g_{\beta\alpha}(p)v = g_{\gamma\alpha}(p)v$, so $(p, \alpha, v) \sim (r, \gamma, x)$. Thus $\sim$ is an equivalence relation.

Let $\hat{E} = \{(p, \alpha, v) \in M \times I \times \mathbb{R}^k : p \in U_\alpha\}$ and define $E = \hat{E}/\sim$. Then on each fiber $E_p = \{[p, \alpha, v] \in E\}$, we define $[p, \alpha, v] + [p, \beta, w] = [p, \alpha, v + g_{\alpha\beta}w]$ and $c[p, \alpha, v] = [p, \alpha, cv]$. Addition is well defined because given any other representation $[p, \alpha', v'] = [p, \alpha, v]$ and $[p, \beta', w'] = [p, \beta, w]$, we see that $[p, \alpha', v'] + [p, \beta', w'] = [p, \alpha', v' + g_{\alpha'\beta'}w'] = [p, \alpha, g_{\alpha\alpha'}v' + g_{\alpha\alpha'}g_{\alpha'\beta'}w'] = [p, \alpha, v + g_{\alpha\alpha'}v' + g_{\alpha\alpha'}g_{\alpha'\beta'}g_{\beta'\beta}w] = [p, \alpha, v + g_{\alpha\beta}w] = [p, \alpha, v] + [p, \beta, w]$. Similarly, if $[p, \alpha', v'] = [p, \alpha, v]$, then $c[p, \alpha', v'] = c[p, \alpha', v']$. So we see that E_p has a well-defined vector space structure.

Question 3.

The map $t_\alpha \circ t_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to (U_\alpha \cap U_\beta) \times \mathbb{R}^n$ by mapping $(p, v) \overset{t_\beta^{-1}}{\mapsto} [p, \beta, v] = [p, \alpha, g_{\alpha\beta}(p)v] \overset{t_\alpha}{\mapsto} (p, g_{\alpha\beta}(p)v)$. Now as $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$ is C^∞ and matrix multiplication in polynomial in its entries (so matrix multiplication is also smooth), we see that $t_\alpha \circ t_\beta^{-1}$ is a smooth map. Similarly we see that $t_\beta \circ t_\alpha^{-1} = (t_\alpha \circ t_\beta^{-1})^{-1}$ is a smooth map (since we can just interchange α and β in the above argument), so $t_\alpha \circ t_\beta^{-1}$ is a diffeomorphism.

Question 4.

From the previous question, $t_\alpha \circ t_\beta^{-1}$ is a diffeomorphism for all α, β. Given any p, we let $p \in U_\alpha$ for some U_α. Let (ϕ_p, V_p) be a chart in M such that $V_p \subset U_\alpha$, and define a map $\Psi_p : \pi^{-1}(V_p) \to \phi(V_p) \times \mathbb{R}^k$ to be the composition $(\phi_p \times Id) \circ t_\alpha$. Note that Ψ_p is a bijective map, and its image is open in $\mathbb{R}^n \times \mathbb{R}^k$. Now given any two such Ψ_p and Ψ_q (where $q \in U_\beta$), if $V_p \cap V_q \neq \emptyset$, we see that $\Psi_p \circ \Psi_q^{-1} = (\phi_p \times Id) \circ t_\alpha \circ t_\beta^{-1} (\phi_q \times Id)^{-1}$. Since $(\phi_p \times Id), t_\alpha \circ t_\beta^{-1}$, and $(\phi_q \times Id)^{-1}$ are C^∞ diffeomorphisms, we see that $\Psi_p \circ \Psi_q^{-1}$ is a C^∞ diffeomorphism. So we first define an open basis on E to be sets of the form $\Psi_p^{-1}(V \times \mathbb{R}^k)$ for some open set $V \in \mathbb{R}^n$. Then with this topology on E, we see that Ψ_p is a homeomorphism to an open subset of $\mathbb{R}^n \times \mathbb{R}^k$ under this topology. The condition that $\Psi_p \circ \Psi_q^{-1}$ is a C^∞ diffeomorphism tells us that $\{(\pi^{-1}(V_p), \Psi_p)\}$ form compatible charts. So taking the maximal atlas containing these, we have a C^∞ structure on E.

The map $\pi : E \to M$ is smooth because in coordinates (ϕ_p, V_p) and $(\Psi_p, \pi^{-1}(V_p))$, the map $\phi_p \circ \pi \circ \Psi_p^{-1}$ maps $(\phi_p^1(a), ..., \phi_p^n(a), v_1, ..., v_k) \mapsto (\phi_p^1(a), ..., \phi_p^n(a))$, which is C^∞. On the other hand, t_α is a smooth map because in coordinates $(\Psi_p, \pi^{-1}(V_p))$ and $(V_p \times \mathbb{R}^k, \phi_p \times Id)$, the map $\phi_p \times Id \circ t_\alpha \circ \Psi_p^{-1}$ is the identity map, which is clearly C^∞. Thus $E \to B$ is a smooth vector bundle with smooth local trivialization t_α.

Question 5.

Suppose $\pi' : E' \to M$ is a bundle that has the same collection of trivialization neighborhoods $\{U_\alpha\}$ and the same transition map such that $E' \to M$ is smooth and t_α are smooth local trivializations, then using the fact that t_α is a diffeomorphism onto its image (verified in the previous part), we see that $t_\alpha^{-1} \circ t_\alpha : E \to E'$ (where the first t_α^{-1} is the map for E and t_α is considered for E') is a diffeomorphism. So we see that locally E and E' are diffeomorphic, and hence E is diffeomorphic to E'. The fact that this diffeomorphism commutes over M (i.e. $\pi = \pi' \circ t_\alpha^{-1} \circ t_\alpha$) follows from the contruction. Thus we see that $E' \to M$ is isomorphic as a vector bundle to $E \to M$.