Math 214 Problem Set 2

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Question 2-20.

Since we are allowed to use Sard's Theorem, we see that given a C^1 mao $f : M^m \to N^n$ where $n > m$, for any point $p \in M^m$, it follows naturally that the rank of f is $< n$ (as the rank is $< m$). Thus all points in M^m are critical points, and so $f(M)$ are all critical values. Sard's theorem states that if $f : M^m \to N^n$ is C^1 and M^m has at most countably many components, then the critical value of f form a set of measure 0 in N^n. Since $f(M)$ are all critical values, $f(M)$ has measure 0 in N^n.

Question 2-28.

Let U_0 be the open set containing $f^{-1}(y)$, then $f|_{U_0}$, which from now on will be denoted just by f, is a map of constant rank k. Now by the "constant rank theorem" proved in the book, we see that for each $p \in f^{-1}(y)$, there exist smooth charts (U, ϕ) centered at p and (V, ψ) centered at $f(p) = y$ such that $\psi \circ f \circ \phi^{-1}(x^1, ..., x^k, x^{k+1}, ..., x^m) = (x^1, ..., x^k, 0, ..., 0)$. Consider $f^{-1}(y) \cap U$ open in $f^{-1}(y)$. $\phi(f^{-1}(y) \cap U) = \{(x^1, ..., x^m) \in U : x^1 = ... = x^k = 0\} = \phi(U) \cap P$ where P is the $(n-k)$-plane $\{x^1 = ... = x^k = 0\}$. Since $P \cong \mathbb{R}^{n-k}$ and $\phi(f^{-1}(y) \cap U)$ is open in P, we see that $(f^{-1}(y) \cap U, \phi)$ is a chart ($(n-k)$-dimensional) for $p \in f^{-1}(y)$. Finally, $f^{-1}(y)$ is closed by continuity, so $f^{-1}(y)$ is a closed submanifold of dimension $n - k$.

Question 2-33a.

First is was noted in the question that the dimension of $GL(n, \mathbb{R})$ is n^2. Consider the determinant function $\det : GL(n, \mathbb{R}) \to \mathbb{R}$, we see that the group

$$\mathrm{SL}(n, \mathbb{R}) = \{M \in GL(n, \mathbb{R}) : \det M = 1\} = \det^{-1}(1)$$

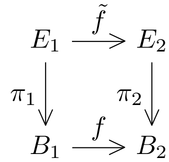
The determinant map is C^1, and the Jacobi's formula states that (I found this under the Wikipedia page "Jacobi's formula")

$$D(\det)_A(B) = (\det A)\mathrm{Tr}(A^{-1}B)$$

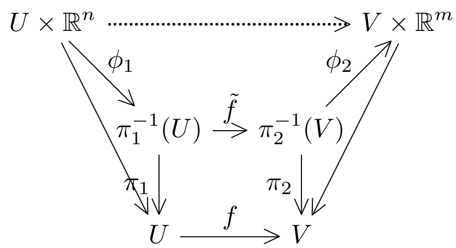
We will verify that $D(\det)_A$ has rank 1 for all A. Consider any arbitrary A, then we want to show that the linear map $D(\det)_A$ (can be seen as a $1 \times n^2$ matrix) is not the zero map, and so it will have rank at least 1, and as the image of det is 1-dimensional, it has rank exactly 1. Now by the formula, $D(\det)_A(A) = (\det A)\mathrm{Tr}(A^{-1}A) = (\det A)\mathrm{Tr}(I) = n * \det A \neq 0$ as $A \in GL(n, \mathbb{R})$, and thus $D(\det)_A$ has rank exactly 1 for any A, and hence $D(\det)$ has constant rank 1. Now by the previous question (2-28), we see that this implies $\mathrm{SL}(n, \mathbb{R}) = \det^{-1}(1)$ is a closed submanifold of $GL(n, \mathbb{R})$ of dimension $n^2 - 1$.

Question 3-4.

Suppose $\tilde{f} : E_1 \to E_2$ is a continuous map and $f : B_1 \to B_2$ is a map (not necessarily continuous) such that the square commutes:



and $\tilde{f} : \pi_1^{-1}(p) \to \pi_2^{-1}(f(p))$ is linear, we will show that f has to be continuous. Now given any open $W \subset B_2$ and any point $p \in f^{-1}(W)$, we can choose, by local triviality, an open $p \in U \subset E_1$ and an open $f(p) \in V \subset W$ to obtain the following diagram:



Now as $\tilde{f}$ is continuous, and ϕ_1, ϕ_2 are homeomorphisms such that the two triangles commute, we see that the map $\phi_2 \circ \tilde{f} \circ \phi_1$ is continuous (shown as the dotted map), and the diagram above (with the dotted arrow) commutes. However, by the commutativity of the diagram, we see that the map $\phi_2 \circ \tilde{f} \circ \phi_1$ is of the form

$$\phi_2 \circ \tilde{f} \circ \phi_1(u, r) = (f(u), \psi(r))$$

for some linear transformation ψ, and as $\phi_2 \circ \tilde{f} \circ \phi_1$ is continuous, we see that there is an open neighborhood $U_0 \times O \subset U \times \mathbb{R}^n$ such that $\phi_2 \circ \tilde{f} \circ \phi_1(U_0 \times O) \subset V \times \mathbb{R}^m$, which means $f(U_0) \subset V \subset W$, and therefore we have proved that given any open $W \in B_2$, $f^{-1}(W)$ is open. Thus f is continuous.

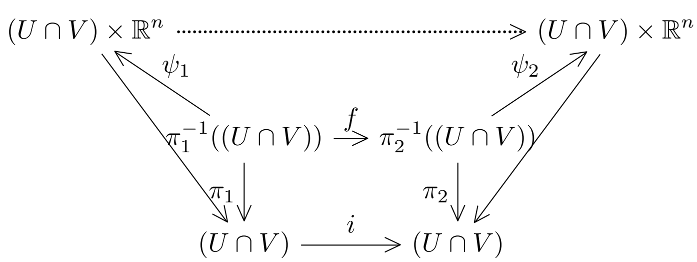
Question 3-7.

(a) Suppose $S : B \to E$ is a map such that $s(p)$ is the 0 vector of $\pi^{-1}(p)$ for any p, we will show that this is a section. First, the condition $\pi \circ s = \text{Id}$ is clear because by our definition, $s(p) \in \pi^{-1}(p)$ for all p. Thus it remains to show that s is continuous. Given any $p \in B$, there exists open $p \in U$ such that $\psi : \pi^{-1}(U) \cong U \times \mathbb{R}^n$ which is a linear isomorphism on its fibers (by local triviality). Consider then $\psi \circ s : U \to U \times \mathbb{R}^n$, we see that $\psi \circ s(p) = \psi(0_p) = (p, 0)$, so $\psi \circ s$ is clearly a continuous function, and thus $s|_U = \psi^{-1} \circ (\psi \circ s)$ is a continuous function. Now as continuity is a "local" property, we conclude that s is continuous, and hence s, the zero section, is actually a section.

(b) We will proceed by first proving the following lemma:

Lemma 1. If $\pi_1 : E_1 \to B$ and $\pi_2 : E_2 \to B$ are two vector bundles over a base B and $f : E_1 \to E_2$ be a continuous map which maps each vector space $(E_1)_p$ isomorphically onto the space $(E_2)_p$, then f is a homeomorphism, and thus the bundles $\pi_1 : E_1 \to B$ and $\pi_2 : E_2 \to B$ are isomorphic.

Proof. Consider $p \in B$, and local trivializations $\psi_1 : \pi_1^{-1}(U) \cong U \times \mathbb{R}^n$ and $\psi_2 : \pi_2^{-1}(V) \cong V \times \mathbb{R}^n$. Then consider $\psi_2 \circ f \circ \psi_1^{-1} : (U \cap V) \times \mathbb{R}^n \to (U \cap V) \times \mathbb{R}^n$, we see, by the commutativity of the diagram



that $\psi_2 \circ f \circ \psi_1^{-1}(u,r) = (u, \xi_u(r))$ where ξ_u is a linear isomorphism $\mathbb{R}^n \to \mathbb{R}^n$ of the fiber above u (because f, ψ_1, and ψ_2 on the fiber above u are all linear isomorphisms) depending continuously on u. Then we see that $\psi_1 \circ f^{-1} \circ \psi_2^{-1}(u,r) = (u, (\xi_u)^{-1}(r))$. Now as taking inverses in $GL(n,\mathbb{R})$ (a map $M \mapsto M^{-1}$) is a continuous function, we see that $(\xi_u)^{-1}$ depends continuously on u. Also, for a normed vector space, for which $\mathbb{R}^n$ is one, linear maps are continuous functions. Thus $(\xi_u)^{-1}(r)$ is continuous on $(U \cap V) \times \mathbb{R}^n$, and so $\psi_1 \circ f^{-1} \circ \psi_2^{-1}$ is continuous. Thus f is a homeomorphism. □

Now suppose $s_1, s_2, ..., s_n$ are everywhere linearly independent, then we define a map $\Psi : B \times \mathbb{R}^n \to E$ by mapping $(p, (a_1, ..., a_m)) \mapsto a_1 s_1(p) + ... + a_n s_n(p)$. Since s_i are all continuous, we see that Ψ is a continuous map, and as $s_i(p)$ are all linearly independent, we see that Ψ maps $p \times \mathbb{R}^n$ isomorphically onto E_p. Thus by the lemma, $B \times \mathbb{R}^n \cong E$ and so $E \to B$ is a trivial bundle.

On the other hand, suppose $\pi : E \to B$ is a trivial bundle, where $\Psi : E \cong B \times \mathbb{R}^n$ is the trivialization, then define $s_i^{'} : B \to B \times \mathbb{R}^n$ to be $s_i^{'}(p) = (p, e_i)$ where i is the standard basis of $\mathbb{R}^n$. Continuity is clear, so we see that $s_i^{'}$ is a section. Now let $s_i = \Psi^{-1} \circ s_i^{'}$, then s_i is a section (continuity follows from composition), and since Ψ is an isomorphism of the fibers, $\{s_i(p)\}_{i=1,...,n}$ are linearly independent for all p. Thus $\{s_i(p)\}_{i=1,...,n}$ are the n sections which are everywhere linearly independent, as desired.

(c) This is just an application of part (b), because locally every vector bundle is trivial, and so that locally every n-plane bundle has n linearly independent sections.

Question 3-19.

(a) I sort of don't understand what this first part is asking. In the question, the division algebra is defined so that it satisfy $a \cdot (1, 0, ..., 0) = a$, and $e_1 = (1, 0, ..., 0)$, so it is clear that for every point $p \in S^{n-1}$, it is equal to $p \cdot e_1$ for some unique p. Hopefully I didn't understand this wrong and made it too trivial.

(b) To show that $a \cdot e_1, a \cdot e_2, ..., a \cdot e_n$ are linearly independent, suppose there exist $k_1, k_2, ..., k_n$ such that $k_1 a \cdot e_1 + k_2 a \cdot e_2 + ... + k_n a \cdot e_n = 0$, then $a \cdot k_1 e_1 + a \cdot k_2 e_2 + ... + a \cdot k_n e_n = 0$, so $a \cdot (k_1 e_1 + \cdot k_2 e_2 + ... + \cdot k_n e_n) = 0$. Since there are no zero divisors, and $a \neq 0$, it must be that $k_1 e_1 + \cdot k_2 e_2 + ... + \cdot k_n e_n = 0$, so $k_1, ..., k_n = 0$ as $e_1, ..., e_n$ are linearly independent. Thus we have shown that $a \cdot e_1, a \cdot e_2, ..., a \cdot e_n$ are linearly independent.

(c) If $p = a \cdot e_1 \in S^{n-1}$, then by the previous part, we see that $a \cdot e_1, a \cdot e_2, ..., a \cdot e_n$ are linearly independent. Now as $(S^{n-1}, i)_p$ consists of all elements of $T_p\mathbb{R}^n$ that is perpendicular to $p = a \cdot e_1$, we see that the projection of $a \cdot e_1, a \cdot e_2, ..., a \cdot e_n$ on $(S^{n-1}, i)_p$, denoted $\overline{a \cdot e_1}, ..., \overline{a \cdot e_n}$ are of the form $\overline{a \cdot e_i} = a \cdot e_i - r_i(a \cdot e_1)$ for some r_i. Therefore, if for some $k_2, ..., k_n$, $k_2 \overline{a \cdot e_2} + ... + k_n \overline{a \cdot e_n} = 0$, then we see that $k_2 a \cdot e_2 + ... + k_n a \cdot e_n - (k_2 r_2 + ... + k_n r_n) a \cdot e_1 = 0$, so by part (b), $k_2 = ... = k_n = 0$, and hence $\overline{a \cdot e_1}, ..., \overline{a \cdot e_n}$ are linearly independent.

(d) We will show that given any e_i the multiplication map $r_{e_i} : \mathbb{R}^n \to \mathbb{R}^n$ by mapping $a \mapsto a \cdot e_i$ is continuous. Note that r_{e_i} is a linear map because the original map $(a, b) \mapsto a \cdot b$ from $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is bilinear. Now as linear maps between finite dimensional normed vector spaces are continuous, it follows that $r_{e_i} : \mathbb{R}^n \to \mathbb{R}^n$ by mapping $a \mapsto a \cdot e_i$ is continuous.

(e) To show that TS^{n-1} is trivial, we construct n sections from $S^{n-1} \to TS^{n-1}$ such that they are everwhere linearly independent. To do this, for any $p \in S^{n-1}$, $p = p \cdot e_1$, so consider the map $s_i : S^{n-1} \to TS^{n-1}$ by taking $p \mapsto \overline{p \cdot e_i} \in (S, i)_p \cong T_p S^{n-1}$, then since the map $a \mapsto a \cdot e_i$ is continuous (by part (d)), and projections to subspaces is continuous (we have also write out explicit formulas in part (c)), we see that s_i is a continuous function. It is also clear that $\pi \circ s = $Id. Thus s_i is a section for all i. Given any $p \in S^{n-1}$, part (c) shows that $\{s_i(p)\}_{i=1,...,n}$ are linearly independent. Thus we have n sections which are everywhere lnearly independent, so by question 3-7(b), TS^{n-1} is trivial.

Question 3-26.

(a) Given any point $p \in M \times N$, we see that a smooth chart in $M \times N$ containing p is of the form

$(U_1 \times U_2, x \times y)$ where $(U_1, x), (U_2, y)$ are smooth charts of M and N respectively. Let $x = (x^1, ..., x^m)$ and $y = (y^1, ..., y^m)$ be the coordinate functions. Thus we see that in coordinates, $T_p(M \times N)$ consists of elements of the form $(p, a_1 \frac{\partial}{\partial x_1}|_p + a_2 \frac{\partial}{\partial x_2}|_p + ... + a_m \frac{\partial}{\partial x_m}|_p + a_{m+1} \frac{\partial}{\partial y_1}|_p + ... + a_{m+n} \frac{\partial}{\partial y_n}|_p)$, and thus $T_p(M \times N)$, by our definition, is equal to the direct sum of the vector spaces $\pi_M^*(TM)_p \oplus \pi_N^*(TN)_p$. Thus we define a map from $\pi_M^*(TM) \oplus \pi_N^*(TN) \to T(M \times N)$ by mapping $((p, a_1 \frac{\partial}{\partial x_1}|_p + a_2 \frac{\partial}{\partial x_2}|_p + ... + a_m \frac{\partial}{\partial x_m}|_p), (p, a_{m+1} \frac{\partial}{\partial y_1}|_p + a_{m+2} \frac{\partial}{\partial y_2}|_p + ... + a_{m+n} \frac{\partial}{\partial y_n}|_p)) \mapsto (p, a_1 \frac{\partial}{\partial x_1}|_p + a_2 \frac{\partial}{\partial x_2}|_p + ... + a_m \frac{\partial}{\partial x_m}|_p + a_{m+1} \frac{\partial}{\partial y_1}|_p + ... + a_{m+n} \frac{\partial}{\partial y_n}|_p)$. This map is continuous, and on every fiber, it is a linear isomorphism. Thus by the lemma we proved in question $3 - 7(b)$, it is a bundle isomorphism.

(b) If M and N are orientable, then using the result from $3 - 23(e)$, we see that $\pi_M^*(TM)$ and $\pi_N^*(TN)$ are both orientable, and $3 - 24(e)$ tells us that $\pi_M^*(TM) \oplus \pi_N^*(TN)$ is orientable. Since $T(M \times N) \cong \pi_M^*(TM) \oplus \pi_N^*(TN)$, we conclude that $M \times N$ is orientable.

(c) On the other hand, if $M \times N$ is orientable, then $\pi_M^*(TM) \oplus \pi_N^*(TN)$ is orientable, and $3 - 24(f)$ tells us that $\pi_M^*(TM)$ and $\pi_N^*(TN)$ are both orientable. Now as the underlying set for the top space of $\pi_M^*(TM)$ is a subset of $M \times N \times TM$, we see that if there is an orientation for $\pi_M^*(TM)$, it means that there is a "compatible" choice of orientation for the fiber above p for all p. But then taking the subset of the top space of $\pi_M^*(TM)$ consisting of a fixed $n \in N$ gives a "compatible" orientation for the bundle TM, and thus TM has an orientation, and therefore M is orientable. Similarly, N is orientable.

Question 1.

As f is differentiable on $\mathbb{R}^3 - (0, 0, z)$, computing $Df|_{(x_0,y_0,z_0)}$ gives

$$Df|_{(x_0,y_0,z_0)} = \left(-2x \left(\frac{2}{\sqrt{x^2 + y^2}} - 1 \right), -2y \left(\frac{2}{\sqrt{x^2 + y^2}} - 1 \right), 2z \right)$$

Consider $f^{-1}(1)$. As $Df|_{(x_0,y_0,z_0)}$ is the zero map at exactly the points $\{(x_0, y_0, 0) \in \mathbb{R}^3 : x_0^2 + y_0^2 = 4\}$, which maps, under f, to 0. Thus as $\mathbb{R}^3$ is normal, there exists an open neighborhood of $f^{-1}(1)$ such that $Df|_{(x_0,y_0,z_0)}$ is not the zero map. Thus on that neighborhood, $Df|_{(x_0,y_0,z_0)}$ has constant rank 1, and thus by question 2-28, we see that $M = f^{-1}(1)$ is a closed submanifold of $\mathbb{R}^3$ of dimension 2, and thus it is a smooth manifold.

Question 2.

$N_r = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = r^2\}$. For $r < 1$ and $r > 3$, $N_r \cap M = \emptyset$, so we can just consider $1 \leq r \leq 3$. For $1 < r < 3$, we see that the intersection is the disjoint union of two circles $\{(x, y, z_0) : x^2 + y^2 = r^2 \text{ and } 1 - (2 - r)^2 = z_0^2\}$, while when $r = 1$ or $r = 3$ the intersection is one single circle of radius r on the x, y-plane, and hence $M \cap N_r$ is a smooth manifold for $1 \leq r \leq 3$. Suppose $1 \leq r \leq 3$ and $p \in N_r \cap M$. We can parametrize M to be $\{((2 + cos(\phi))cos(\theta), (2 + cos(\phi))sin(\theta), sin(\phi)) : \psi \in [0, 2\pi), \theta \in [0, 2\pi)\}$, and we let $p = ((2 + cos(\phi_0))cos(\theta_0), (2 + cos(\phi_0))sin(\theta_0), sin(\phi_0))$. Then T_pM, considered as embedded in $\mathbb{R}^3$, is spanned by the vectors $p + (-(2 + cos(\phi_0))sin(\theta_0), (2 + cos(\phi_0))cos(\theta_0), 0)$ and $p + (-sin(\phi_0)cos(\theta_0), -sin(\phi_0)sin(\theta_0), cos(\phi_0))$, while T_pN_r is spanned by the vectors $p + (-(2 + cos(\phi_0))sin(\theta_0), (2+cos(\phi_0))cos(\theta_0), 0)$ and $p+(0, 0, 1)$. Thus when $r \neq 1$ and $r \neq 3$, we see that $\phi_0 \neq 0$ or π, and hence $(0, 0, 1)$ and $(-sin(\phi_0)cos(\theta_0), -sin(\phi_0)sin(\theta_0), cos(\phi_0))$ are linearly independent. Therefore, when $1 < r < 3$, $T_p\mathbb{R}^3 = T_pM + T_pN_r$ and so $M \cap N_r$ transversely.

Question 3.

Let $P_h - \{(h, y, z) \in \mathbb{R}^3\}$. Then for $-3 \leq h \leq 3$, $P_h \cap M$ is not empty, and is an embedded submanifold of M almost directly by definition of an embedded submanifold, because $P_h \cap M$ is an x-slice. We want to find where they intersect transversely. Let $p \in P_h \cap M$. Again we parametrize M to be $\{((2+cos(\phi))cos(\theta), (2+cos(\phi))sin(\theta), sin(\phi)) : \psi \in [0, 2\pi), \theta \in [0, 2\pi)\}$, then T_pM, considered as embedded in $\mathbb{R}^3$, is spanned by the vectors $p+(-(2+cos(\phi_0))sin(\theta_0), (2+cos(\phi_0))cos(\theta_0), 0)$ and $p+(-sin(\phi_0)cos(\theta_0), -sin(\phi_0)sin(\theta_0), cos(\phi_0))$, while T_pP_h is spanned by the vectors $p + (0, 1, 0)$ and $p + (0, 0, 1)$. Therefore, for $-3 < h < 3$, we see that $(-sin(\phi_0)cos(\theta_0), -sin(\phi_0)sin(\theta_0), cos(\phi_0))$ is not in the plane spanned by $(0, 1, 0)$ and $(0, 0, 1)$, and thus $T_p\mathbb{R}^3 = T_pM + T_pP_h$ and so $M \cap P_h$ transversely. On the other hand, if $h = \pm 3$, then the vector listed for T_pM and T_PP_h span the same plane, and so they do not intersect transversely anymore. So we conclude that for $-3 \leq h \leq 3$ exactly, $T_p\mathbb{R}^3 = T_pM + T_pP_h$ and so $M \cap P_h$ transversely.