Math 214 Problem Set 1

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September 16, 2011

Question 1-10.

To show that the limit of a convergent subsequence of the z_i is actually in the closed ball of radius $r(y)$ centered at y, we first denote the subsequence to be z_{n_j}, and denote its limit point to which z_{n_j} converges as z. By the original contruction, we see that z_{n_j} is in the ball of radius $r(y_{n_j})$ centered at y_{n_j}. Since $y_{n_j} \to y$ and r is a continuous function, so given $\epsilon > 0$, for n_j large enough, we see that

$$\begin{aligned} d(z,y) &\leq d(z, z_{n_j}) + d(z_{n_j}, y_{n_j}) + d(y_{n_j}, y) \\ &\leq \epsilon + r(y_{n_j}) + \epsilon \\ &\leq \epsilon + r(y) + \epsilon + \epsilon \\ &= r(y) + 3\epsilon \end{aligned}$$

Thus as ϵ is arbitrary, we see that $d(z, y) \leq r(y)$ and hence the limit of a convergent subsequence of the z_i is actually in the closed ball of radius $r(y)$ centered at y.

Question 1-18.

(a) Let $n > 1$, and let $C \subset \mathbb{R}^n$. Since compact subsets of $\mathbb{R}^n$ are closed and bounded, we see that there exist a closed ball $\bar{B}$ centered at 0 such that $C \subset \bar{B} \subset \mathbb{R}^n$. Now $\mathbb{R}^n - \bar{B}$ is connected (here we are using the Jordan-Brouwer separation theorem), and thus $\mathbb{R}^n$ has one end.

On the other hand, consider $n = 1$ and a compact $C \subset \mathbb{R}$. Given any compact K such that $C \subset K \subset \mathbb{R}$, we know that K is closed and bounded. Let $a = \sup K$ and let $b = \inf K$, then we see that (a, ∞) and $(-\infty, b)$ are two connected components of $\mathbb{R} - K$, and thus $\mathbb{R}$ is not connected. Thus $\mathbb{R}$ does not have one end.

(b) Consider $\mathbb{R}^n - \{0\}$. Let C be the compact set $\overline{B_2(0)} - \overline{B_1(0)}$, then for any compact set K such that $C \subset K \subset \mathbb{R}^n$, we see that $(\mathbb{R}^n - \{0\}) - K$ contains at least two components, and thus $\mathbb{R}^n - \{0\}$ does not have one end, and therefore $\mathbb{R}^n - \{0\}$ is not homeomorphic to $\mathbb{R}^n$ as "has one end" is a topological property (since connectedness and compactness are both topologocal properties).

Question 1-19(a,b).

(a) Given any compact C_1 and C_2, we see first that $\epsilon(C_1)$ and $\epsilon(C_2)$ must be either the left unbounded component or the right unbounded component. Suppose we pick arbitrarily that $\epsilon(C_1)$ is the left unbounded component and $\epsilon(C_2)$ is the right unbounded componen, then consider $C_1 \cup C_2$, which is a compact set containing both C_1 and C_2. So if we pick $\epsilon(C_1 \cup C_2)$ to be the left unbounded component, then $\epsilon(C_1 \cup C_2)$ is not contained in $\epsilon(C_2)$, failing the condition stated. If we pick $\epsilon(C_1 \cup C_2)$ to be the right unbounded component, then $\epsilon(C_1 \cup C_2)$ is not contained in $\epsilon(C_1)$, again failing the condition stated. Since C_1, C_2 are arbitrary, it must be that $\epsilon(C)$ is either always "right" or always "left", and thus $\mathbb{R}$ has 2 ends.

(b) We will do the general case. Suppose X has exactly one end ϵ, then we see that given a compact set C, if for all compact K containing C, $X - K$ is not connected, then there are at least two components for all $X - K$. Then we see that there will be at least two ϵ's such that the "inclusion reversing" criterion is satisfied (we can consistently choose $\epsilon(K)$ to be some component contained in $\epsilon(C)$, and there are at least two choices of $\epsilon(C)$). But this contradicts the fact that X has exactly one end, and so there must be a compact K containing C such that $X - K$ is connected. Thus X "has one end" as defined in the previous problem.

On the other hand, suppose X "has one end" as defined in the previous problem. Then given any C, there is a compact K such that $C \subset K$ and $X - K$ is connected. Then the obvious choice of $\epsilon(C)$ is the connected component of $X - C$ that contains $X - K$. This gives a "consistent" (in the sense that it satisfies the "inclusion reversing" property) choice of ϵ, so we see that X has at least one end. Furthermore, the property that $C \subset K$ implies $\epsilon(K) \subset \epsilon(C)$ implies that there is only one connected component of C that one can pick. Thus $\epsilon(C)$ is unique for all C, and thus X has exactly one end.

Apply this general result to $X = \mathbb{R}^n$ (in combination with the result in the previous problem), we see that $\mathbb{R}^n$ has only one end when $n > 1$.

Question 1-25.

(a) We will show that M, under this alternative definition, satisfies the original definition of being a manifold-with-boundary. Suppose M is a manifold-with-boundary as defined in this question, and $x \in M$ is any point, then there is a neighborhood U of x and an integer $n \geq 0$ such that U is homeomorphic (under a map h) to an open subset $h(U)$ of $\mathbb{H}^n$. Let $h(x) = (x_1, x_2, ..., x_n)$, there are two cases: first, that $x_n = 0$, and second, that $x_n > 0$. If $x_n = 0$, then we see that there is a open half ball (an intersection of an open ball with $\mathbb{H}^n$), denote by $HB_{h(x)}$, centered around $h(x)$ such that $HB_{h(x)} \subset h(U)$. Thus taking $h^{-1}(HB_{h(x)})$ we see that this is open in U and containing x, and we take this to be the neighborhood for the "original definition" given in book. Now as open half balls are homoemorphic to $\mathbb{H}^n$ under a homeomorphism h', we let $h' \circ h$ be the homeomorphism for the "original definition" given in book. Then it is clear that $(h' \circ h, h^{-1}(HB_{h(x)}))$ is a pair such that $h^{-1}(HB_{h(x)})$ is an open neighborhood of x and $h' \circ h$ is a homeomorphism from $h^{-1}(HB_{h(x)})$ to $\mathbb{H}^n$.

On the other hand, if $x_n > 0$, then we see that there is an open ball, denote by $B_{h(x)}$, centered around $h(x)$ such that $B_{h(x)} \subset h(U)$. Thus taking $h^{-1}(B_{h(x)})$ we see that this is open in U and containing x, and we take this to be the neighborhood for the "original definition" given in book. Now as open balls are homoemorphic to $\mathbb{R}^n$ under a homeomorphism h', we let $h' \circ h$ be the homeomorphism for the "original definition" given in book. It is again clear that $(h' \circ h, h^{-1}(B_{h(x)}))$ is a pair such that $h^{-1}(B_{h(x)})$ is an open neighborhood of x and $h' \circ h$ is a homeomorphism from $h^{-1}(B_{h(x)})$ to $\mathbb{R}^n$. Thus we have shown that this definition of manifold M is also a manifold under the original book definition.

(b) Let M be a manifold with boundary. We will show that ∂M is a manifold. Given a point $x \in \partial M$, we see by the first part that there is a U containing x and a homeomorphism h such that $h(U)$ is an open subset of $\mathbb{H}^n$. Now let $V \subset h(U)$ be the set of points such that the last component is 0, then we see that V is an open subset of $\mathbb{R}^{n-1}$ (the subspace of $\mathbb{R}^n$ with the last component 0), and thus $h^{-1}(V)$ is an open subset of ∂M containing x. Furthermore, $h|_{(U \cap \partial M)}$ is a homeomorphism from $h^{-1}(V)$ to V, i.e. a homeomorphism from $h^{-1}(V)$ to an open subset of $\mathbb{R}^{n-1}$, and thus ∂M is a manifold of "dimension" $n - 1$.

On the other hand, consider $M - \partial M$, then given a point $x \in M - \partial M$, by the first part of this quesiton we see that we can always take U to be a neighborhood not intersecting the boundary, and a homeomorphism h which maps U to an open ball in $\mathbb{R}^n$. But this is the definition of being a manifold, and so we conclude that $M - \partial M$ is a manifold (of "dimension" n).

(c) Consider $X = M - \cup_{i \in I'} C_i$. Given any point x in X, we treated first of all as a point in M. If $x \notin \cup_{i \notin I'} C_i$ (i.e. x is not a boundary point in the original M), then a chart of M containing x that map a neighborhood U homoemorphically onto $\mathbb{R}^n$ will also map this neighborhood U homoemorphically onto $\mathbb{R}^n$

when considered in X. Similarly, if $x \in \cup_{i \notin I'} C_i$, then a chart of M containing x that map a neighborhood U' homoemorphically onto $\mathbb{H}^n$ will also map this neighborhood U' homoemorphically onto $\mathbb{H}^n$ when considered in X. Thus X is a manifold-with-boundary.

Question 2-1.

(a) Being C^∞-related is not an equivalence relation, as it fails transitivity. Suppose (x, U) and (y, V) are C^∞-related, and (y, V) and (z, W) are C^∞-related, then it is only guaranteed that (x, U) and (z, W) are C^∞-related on the domain $x(U \cap V \cap W)$ and $z(U \cap V \cap W)$, but says nothing about $x \circ z^{-1}$ nor $z \circ x^{-1}$ on the correct domain $z(U \cap W)$ and $x(U \cap W)$ respectively.

(b) The charts in A' are all charts y' which are C^∞-related to all charts $x \in A$. Thus given any $(f, U_f), (g, U_g) \in A'$, we see that for all $(x, U_x) \in A$, $f \circ x^{-1} : x(U_x \cap U_f) \to f(U_x \cap U_f)$ and $x \circ f^{-1} : f(U_x \cap U_f) \to x(U_x \cap U_f)$ are both C^∞, and similarly for the cases $g \circ x^{-1}$ and $x \circ g^{-1}$. Thus the maps $(f \circ g^{-1})|_{g(U_x \cap U_f \cap U_g)} : g(U_x \cap U_f \cap U_g) \to f(U_x \cap U_f \cap U_g))$ and $(g \circ f^{-1})|_{f(U_x \cap U_f \cap U_g)} : f(U_x \cap U_f \cap U_g) \to g(U_x \cap U_f \cap U_g))$ are both C^∞. Now as $x \in A$ is arbitrary, and as A is an atlas of charts covering M, we apply the above procedure to all $x \in A$, which, by the glueing lemma of smooth functions, allows us to conclude that $(f \circ g^{-1}) : g(U_f \cap U_g) \to f(U_f \cap U_g))$ and $(g \circ f^{-1}) : f(U_f \cap U_g) \to g(U_f \cap U_g))$ are C^∞. Thus $f, g \in A'$ are C^∞-related. This shows that all charts of A' are C^∞-related.

Question 2-4.

I was unable to classify and list out all "distinct" C^∞ structures on $\mathbb{R}$, but I tried to see how "big" the size of this set is. We will do this by first constructing homeomorphisms from $\mathbb{R} \to \mathbb{R}$. Given any real number $r \in \mathbb{R}$, we consider the function

$$f_r(x) = \begin{cases} x^r & \text{if } x \geq 0 \\ -(-x)^r & \text{if } x < 0 \end{cases}$$

The function f_r is continuous and one to one from $\mathbb{R}$ onto $\mathbb{R}$, and hence it is a homeomorphism (as we are in $\mathbb{R}$, f_r is an open mapping by the invariance of domain).

Now consider any two distinct real numbers r and s, we define an altas $\mathcal{A}_r$ and $\mathcal{A}_s$ to be the unique maximal atlas containing $(f_r, \mathbb{R})$ and $(f_s, \mathbb{R})$ respectively, we see that $f_r \circ f_s$ or $f_s \circ f_r$ (or both) cannot be smooth, so we know that $\mathcal{A}_r$ and $\mathcal{A}_s$ must be "distinct". Thus this shows that there are at least $2^{\aleph_0}$ distinct C^∞ structures. On the other hand, the set of continuous functions from $\mathbb{R} \to \mathbb{R}$ have $2^{2^{\aleph_0}}$ size, so we know an upper bound. Thus we have an idea on how many distinct C^∞ structures are on $\mathbb{R}$.

Question 2-14.

(a) Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x, y) = \begin{cases} (x, y^2) & \text{if } y \geq 0 \\ (x, y^3) & \text{if } y < 0 \end{cases}$$

then we see that $f(x, 0 = (x, 0))$ for all x, $f(x, y) \subset \mathbb{H}^2$ for $y \geq 0$, and $f(x, y) \subset \mathbb{R} - \mathbb{H}$ for $y < 0$. This f, when restricted to the upper-helf plane or the lower half plane is C^∞ as all partials of all orders exist and are continuous. The function f, though, is not C^∞ as it is not even C^2. Thus we have shown that such function exists.

(b) We will show that this construction does not define a C^∞ structure on P. Suppose (x, U) and (y, V) are coordinate systems around p and $f(p)$ respectively satisfying the properties described in the problem.

Now consider the coordinate systems $(g \circ x, U)$ and $(r \circ g \circ r \circ y), V)$ where g is the function that is described in part (a) and r is the reflection along the last component. We know that $(g \circ x, U)$ and (x, U) are C^∞ related because $g|_{\mathbb{H}^n}$ and $g^{-1}|_{\mathbb{H}^n}$ are both C^∞. Similarly $(r \circ g \circ r \circ y, V)$ and (y, V) are C^∞ related. Now we use (x, U) and (y, V) to build a chart (W_1, ϕ_1) of $p \in P$, and we use $(g \circ x, U)$ and $(r \circ g \circ r \circ y, V)$ to build a second chart (W_2, ϕ_2) of $p \in P$, then we see that $\phi_2 \circ \phi_1^{-1}$ is precisely the map $g|_{W_1}$, which is not C^∞ by the first part of this question.

(c) We will define a C^∞ structure on P such that the inclusions of M and N are C^∞ and such that the map from $U \cup V$ to $\partial M \times (-1, 1)$ induced by α and β is a diffeomorphism.

$P = M \cup_f N$, we let i_M, i_N be the inclusion map from M, N into P respectively, and let $p \in P$. Suppose $p = i_M(x)$ for some x not on the boundary of M, and suppose (U_0, ϕ) is a chart of $x \in M$, then we can just set the chart of $p \in P$ to be $(i_M(U_0), \phi')$ where $\phi'(a) = \phi \circ i^{-1}(a)$ for any $a \in i_M(U_0)$. Similarly we can do this for $y \in N$ not on the boundary of N.

Now suppose $p \in P$ is in the image of $i_M(x) = i_N \circ f(x)$ where x is on the boundary of M. Recall that $\alpha : U \cong \partial M \times [0, 1)$ and $\beta : V \cong \partial N \times [0, 1)$ are diffeomorphisms. Now we let $[0, 1)$ be the neighborhood and $i : [0, 1) \to [0, 1)$ by the identity chart of the manifold with boundary $[0, 1)$. Given $x \in \partial M$, we proved in question 1-25 that ∂M is a manifold, so let (ψ, U_0) be a chart, we consider the chart $((\psi \times i) \circ \alpha, U_1)$ in M where $\alpha(U_1) = U_0 \times [0, 1)$. We do this similarly for $f(x)$ on the boundary of N to obtain a chart $((\phi \times i) \circ \beta, V_1)$. Now we apply the construction in part (b) to get a chart of p. We see that these charts are C^∞ related as follows: By construction, it is clear that a chart in the interior of M are C^∞ to all the other charts in P. We need only consider two charts at two points p_1, p_2 on the image of the boundary in P. Lets say $(\zeta_1, W_1), (\zeta_2, W_2)$ are two said charts, then $\zeta_1 \circ \zeta_2^{-1}$ is $\eta_1 \circ \eta_2^{-1} \times Id$, where η are the restrictions of the charts to a chart of the boundary, and thus the ζ's are C^{infty} related. We take the maximal atlas containing these charts, and uniqueness follows.

It is clear that the inclusion of M into P is C^∞ at any point x in the interior of M is C^∞. So we consider a point $x \in \partial M$. Then let (ζ, W) be the chart of $i(x)$ in P and let (ψ, U_0) be the chart in M that we construct (ζ, W) from (as described in the previous paragraph). Then $\zeta \circ i \circ \psi^{-1}$ is indeed the identity function, so is C^∞. Similarly we can do this for the inclusion of N into P.

Finally we consider the map Ψ from $U \cup V$ to $\partial M \times (-1, 1)$ induced by α and β. By our construction, a chart in $U \cup V$ comes from gluing charts of the form $((\psi \times i) \circ \alpha, U_1)$ and $((\phi \times i) \circ \beta, V_1)$, which gives a chart $((\phi \circ \beta_1) \times i, U_1 \cup_f V_1)$ on $U \cup V$. However, this chart is precisely $\Psi \circ (\phi \times i)$, and thus we see that the chart in $U \cup V$ corresponds bijectively to charts of $\partial M \times (-1, 1)$ under the map Ψ, and thus Ψ is a diffeomorphism.

(d) Consider the pair (i_1, i_2) of two identity functions on the two copies of $\mathbb{H}^2$, and another pair (f_1, f_2) where $f_1(x, y) = f_2(x, y) = (x, y^2)$ defined on the neighborhood $U = \{(x, y) : 0 \leq y < 1\}$. Then the resulting C^∞ structure on $\mathbb{R}^2$ are diffeomorphic under the map

$$\xi(x) = \begin{cases} x & \text{if } x \geq 1 \\ x^2 & \text{if } 0 \leq x < 1 \\ -x^2 & \text{if } -1 < x < 0 \\ x & \text{if } x \leq -1 \end{cases}$$

and from the explicit form of ξ it is clear that it cannot be chosen arbitrarily close to the identity map.

Question 2-23.

Here I will use results from the four problems below. Assum $n > 1$. Suppose a curve $\gamma : [0, 1] \to \mathbb{R}^n$ is rectifiable, then by definition, $H^1(\gamma) = l(\gamma) < \infty$. Then the measure in $\mathbb{R}^n$, which is $H^n(\gamma)$, is 0 by Question 2 (which is proved below). Thus the image of a rectifiable curve has measure 0.

Question 1.

Assuming the Jordan-Brouwer separation theorem, we suppose $U \subset \mathbb{R}^n$ is open, and $f : U \to \mathbb{R}$ is one to one and continuous. Then given any point $y \in f(U)$, we consider $f^{-1}(y)$. Since $f^{-1}(y) \in U$, we consider a open ball $B_r(f^{-1}(y)) \subset U$ of diameter r centered at $f^{-1}(y)$. Then consider $\overline{B_r(f^{-1}(y))} \subset U$, we first see that, $f(\overline{B_r(f^{-1}(y))})$ is compact, so it is closed and bounded. Thus by the Jordan-Brouwer separation theorem, we see that $f(B_r(f^{-1}(y)))$ is the bounded component, and as connected components are both open and closed in $\mathbb{R}^n - f(\partial \overline{B_r(f^{-1}(y))})$, and we see that $\mathbb{R}^n - f(\partial \overline{B_r(f^{-1}(y))})$ is open in $\mathbb{R}^n$, we see that $f(B_r(f^{-1}(y)))$ is an open set containing y. Thus $f(U)$ is open, and we have proved the invariance of domain.

Question 2.

Let A be a subset of a metric space (X, d) and $n, m \in (0, \infty)$ with $n > m$. Suppose $\mathcal{H}^m(\mathcal{A}) = k < \infty$, then by definition $\lim_{\delta \to 0^+} \phi_{m,\delta}(A) = k < \infty$. Consider $\phi_{n,\delta}(A)$ given any fixed δ, we see that

$$\begin{aligned}
\phi_{n,\delta}(A) &= \inf_G \sum_{S \in G} \zeta_n(S) \\
&\leq inf_G \; \delta^{n-m} \frac{\alpha(n)}{\alpha(m)} \sum_{S \in G} \zeta_m(S) \\
&= \delta^{n-m} \frac{\alpha(n)}{\alpha(m)} inf_G \sum_{S \in G} \zeta_m(S) \\
&= \delta^{n-m} \frac{\alpha(n)}{\alpha(m)} \phi_{m,\delta}(A)
\end{aligned}$$

So we see that

$$\lim_{\delta \to 0^+} \phi_{n,\delta}(A) \leq \lim_{\delta \to 0^+} \delta^{n-m} \frac{\alpha(n)}{\alpha(m)} \phi_{m,\delta}(A) = \lim_{\delta \to 0^+} \left(\delta^{n-m} \frac{\alpha(n)}{\alpha(m)} \right) \left(\lim_{\delta \to 0^+} \phi_{m,\delta}(A) \right) = 0 * k = 0$$

Thus $\mathcal{H}^n(\mathcal{A}) = 0$

Question 3.

Let A be a subset of a metric space (X, d) and $n, m \in (0, \infty)$ with $n > m$. Suppose $\mathcal{H}^n(\mathcal{A}) = k > 0$. By definition $\lim_{\delta \to 0^+} \phi_{n,\delta}(A) = k > 0$. Consider $\phi_{m,\delta}(A)$ given any fixed δ, we see that

$$\begin{aligned}
\phi_{m,\delta}(A) &= \inf_G \sum_{S \in G} \zeta_m(S) \\
&\geq inf_G \; \delta^{m-n} \frac{\alpha(m)}{\alpha(n)} \sum_{S \in G} \zeta_n(S) \\
&= \delta^{m-n} \frac{\alpha(m)}{\alpha(n)} inf_G \sum_{S \in G} \zeta_n(S) \\
&= \delta^{m-n} \frac{\alpha(m)}{\alpha(n)} \phi_{n,\delta}(A)
\end{aligned}$$

So we see that

$$\lim_{\delta \to 0^+} \phi_{m,\delta}(A) \geq \lim_{\delta \to 0^+} \delta^{m-n} \frac{\alpha(m)}{\alpha(n)} \phi_{n,\delta}(A) = \infty$$

Thus $\mathcal{H}^m(\mathcal{A}) = \infty$

Question 4.

Suppose (X, d) is a metric space and $\alpha \in (0, 1)$, we consider the metric space (X, d^α). Suppose there is some rectifiable curve γ under the metric d^α, then what this translates to in terms of Hausdorff dimension is that $H^\alpha(\gamma) < \infty$ in the space (X, d). However, by Question 2, we see that this implies that $H^1(\gamma) = l(\gamma) = 0$ in the space (X, d), so in the space (X, d) this curve γ is trivial, and hence in the space (X, d^α) it must be the trivial curve as well. Thus we conclude that (X, d^α) is a metric space with no nontrivial rectifiable curves.