214 - Homework Assignment # 1

Complete the following exercises in Spivak:

1-10, 1-18, 1-19(a,b), 1-25, 2-1, 2-4, 2-14, 2-23.

For 1-19b, assume X is a connected manifold. Feel free to use the Axiom of Choice if needed in your proof. (If you are not familiar with the Axiom of Choice, you might end up using it anyway. Don't worry about this unless you want to.)

Additionally, complete the following four problems. Relevant definitions and background material follow the problem statements.

1. Prove Invariance of Domain, assuming the Jordan-Brouwer separation theorem.
2. Let A be a subset of a metric space (X, d) and $n, m \in (0, \infty)$ with $n > m$. Prove that $\mathcal{H}^m(A) < \infty$ implies $\mathcal{H}^n(A) = 0$.
3. Let A be a subset of a metric space (X, d) and $n, m \in (0, \infty)$ with $n > m$. Prove that $\mathcal{H}^n(A) > 0$ implies $\mathcal{H}^m(A) = \infty$.
4. Let (X, d) be a metric space and $\alpha \in (0, 1)$. Prove that (X, d^α) is a metric space with no nontrivial rectifiable curves.

Theorem 1 (Jordan-Brouwer separation theorem)**.** *If $f : D^n \to \mathbb{R}^n$ is injective and continuous, then $\mathbb{R}^n \setminus f(\partial D^n)$ has two components: $\mathbb{R}^n \setminus f(D^n)$ and $f(\operatorname{int} D^n)$. Furthermore, $\mathbb{R}^n \setminus f(D^n)$ is unbounded and $f(\operatorname{int} D^n)$ is bounded.*

The remaining discussion will define the Hausdorff measure $\mathcal{H}^m$, providing context for Spivak's use of "measure zero" for anyone unfamiliar with Lebesgue measure.

Recall the method of definition of arc-length for a curve $\gamma : [a, b] \to \mathbb{R}^n$ from calculus. There, we underestimate the length by taking a partition $a = t_1 < t_2 < \cdots < t_k = b$ and computing

$$\sum_{i=1}^{k-1} d(\gamma(t_i), \gamma(t_{i+1})).$$

The length of γ is then defined to be the supremum of all such underestimates over all partitions of $[a, b]$. (For C^1 curves, the length is equivalently the limit of these underestimates as the mesh of the partition goes to zero, leading to the familiar integration formula.) Everything here makes sense in a general metric space (X, d), leading to the following definition.

Definition 1. For a curve $\gamma : [a, b] \to (X, d)$, we define the length $l(\gamma)$ of γ to be

$$l(\gamma) = \sup_{a=t_1<t_2<\cdots<t_k=b} \sum_{i=1}^{k-1} d(\gamma(t_i), \gamma(t_{i+1})).$$

If $l(\gamma) < \infty$, then γ is called a rectifiable curve.

Of course, for objects more complicated than curves and areas or higher dimensional volumes, the situation will be trickier. What follows is one useful generalization of the previous concept.

We will be defining a notion of m-dimensional volume of a set A in a metric space (X, d). For curves, we estimated the length of an arc of the curve by taking $d(\gamma(t_i), \gamma(t_{i+1}))$. We need a comparable m-dimensional notion for a "piece" of A. Our choice will be to look at $\operatorname{diam} S$ for subsets S of A, where

$$\operatorname{diam} S = \sup_{x,y\in S} d(x, y).$$

If (X, d) is Euclidean $\mathbb{R}^m$ $(m \neq 1)$ and $S = B_r(0)$, then $\operatorname{diam} B_r(0)$ does not grow like the volume as r varies, so we actually want $(\operatorname{diam} S)^m$ in order to get something close to volume. We now have a notion that matches volumes for balls, up to a proportional constant depending on m, so we set

$$\zeta_m(S) = \alpha(m)(\operatorname{diam} S)^m,$$

where $\alpha(m)$ is to be determined later. (The precise value of $\alpha(m)$ does not matter much in most applications and you should be able to ignore it in the problems.) Intuitively, computing $\zeta_m(S)$ for a set of large diameter is like measuring with a ruler that only has large distances marked, so restricting $\operatorname{diam} S$ should give us a better estimate. We therefore introduce

$$\phi_{m,\delta}(A) = \inf \sum_{S\in G} \zeta_m(S),$$

where the infimum is taken over $G \subset \{S : \operatorname{diam} S \leq \delta\}$ with G covering A. The construction trivially gives that $\phi_{m,\delta}(A)$ is monotone in δ, allowing us to guarantee existence of the limit in the following definition.

Definition 2. The m-dimensional Hausdorff measure $\mathcal{H}^m(A)$ of $A \subset (X, d)$ is

$$\mathcal{H}^m(A) = \lim_{\delta\to 0^+} \phi_{m,\delta}(A) = \sup_{\delta>0} \phi_{m,\delta}(A).$$

We always have $\mathcal{H}^m(A) \in [0, \infty]$. In fact, Hausdorff measure is an honest measure, or more precisely a Borel regular measure, if you are familiar with those terms. You will show that for a fixed A, $\mathcal{H}^m(A) \in (0, \infty)$ for at most one value of m, giving us a notion of dimension for A, called the Hausdorff dimension. One interesting property of this definition is that we never used an assumption that m is an integer, allowing for the existence of sets of fractional (or even irrational) Hausdorff dimension!

As a final step, we need to define $\alpha(m)$. Again, the precise value of this tends not to matter much, so further reading should only be of your own volition. For $m \in \mathbb{N}$, we know we want $\alpha(m)$ to force ζ_m to give the correct value for balls in $\mathbb{R}^m$. Unless your are particularly attached to closed-form expressions, we may simple declare

$$\alpha(m) = \frac{\mathrm{Vol}\{x \in \mathbb{R}^m : \|x\| < 1\}}{2^m}$$

and be done with the integer case. For non-integral $m > 0$ however, this makes no sense. We need a closed-form expression that gives the correct values on $\mathbb{N}$ and is defined for all nonnegative numbers. The choice taken for Hausdorff measure is

$$\alpha(m) = \frac{(\Gamma(1/2))^m}{2^m \Gamma(1 + (m/2))}$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$$

is the usual Gamma function. This is mere convention, as any α matching the correct values for $\mathbb{N}$ will still give the same relationships to classical volumes. Furthermore, another choice would simply correspond to a change in units of measurement. We therefore use the above α as it is a particularly clean choice.