Math 53 - Multivariable Calculus

Homework # 4

July 19th
Due: July 24th, 2012

Solns

Exercise 1 (10 points). *Let $\vec{F}(x,y) = r^n(x\hat{\imath} + y\hat{\jmath})$, where $r = \sqrt{x^2+y^2}$*

(a) For which values of n do the components $P = r^n x$ and $Q = r^n y$ of $\vec{F}$ satisfy $\partial P/\partial y = \partial Q/\partial x$? (Hint: start by finding formulae for r_x and r_y).

(b) Whenever possible, find a function g such that $\vec{F} = \nabla g$. (Hint: look for a function of the form $g = g(r)$, with $r = \sqrt{x^2+y^2}$.)

(a) If $\vec{F} = \langle r^n x, r^n y \rangle \Rightarrow Q_x = \frac{\partial}{\partial x}(yr^n) = nyr^{n-1}\frac{x}{r}$

and $P_y = \frac{\partial}{\partial y}(xr^n) = nxr^{n-1}\frac{y}{r}$ since $r_x = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$ and

$r_y = \frac{y}{r}$. Hence $Q_x = P_y$.

(b) If $g = g(r) \Rightarrow g_x = \frac{dg}{dr}\frac{\partial r}{\partial x} = g'(r)\frac{x}{r}$ and $g_y = g'(r)\frac{y}{r}$.

So, $\nabla g = \langle g'(r)\frac{x}{r}, g'(r)\frac{y}{r} \rangle = \frac{g'(r)}{r}\langle x, y \rangle$. Thus, we

must find g such that $\frac{g'(r)}{r} = r^n$; i.e., $g'(r) = r^{n+1}$. There

are two cases:

$n \neq -2$	$n = -2$
$g(r) = \frac{1}{n+2}r^{n+2}$	$g(r) = \ln(r)$

Exercise 2 (10 points). *Let* $\vec{F}(x,y) = \dfrac{-y\hat{\imath} + x\hat{\jmath}}{x^2+y^2}$

(a) Show that $\vec{F}$ is the gradient of the polar angle function $\theta(x,y) = \tan^{-1}(y/x)$ defined over the right half-plane $x > 0$.

(b) Suppose that C is a smooth curve in the right half-plane $x > 0$ joining two points $A : (x_1, y_1)$ and $B : (x_2, y_2)$. Express $\int_C \vec{F} \cdot d\vec{r}$ in terms of the polar coordinates (r_1, θ_1) and (r_2, θ_2) of A and B.

(c) Compute directly from the definition the line integrals $\int_{C_1} \vec{F} \cdot d\vec{r}$ and $\int_{C_2} \vec{F} \cdot d\vec{r}$, where C_1 is the upper half of the unit circle running from $(1,0)$ to $(-1,0)$, and C_2 is the lower half of the unit cirlce, also running from $(1,0)$ to $(-1,0)$.

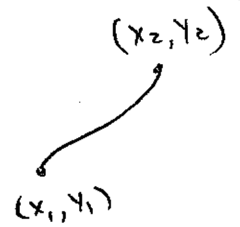
(d) Using the results of parts (a)-(c), if $\vec{F}$ conservative (path-independent) over its entire domain of definition? Is it conservative over the right half-plane $x > 0$? Be sure to justify your answers.

(e) Show that the components $P = -y/(x^2+y^2)$ and $Q = x/(x^2+y^2)$ of $\vec{F}$ satisfy the equation $\partial P/\partial y = \partial Q/\partial x$ at any point of the plane where $\vec{F}$ is defined (not just in the right half-plane $x > 0$).

(f) Show that $\int_C \vec{F} \cdot d\vec{r} = 0$ for every simple closed curve that does not pass through or enclose the origin.

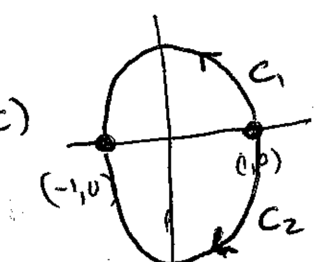
(a) For $\theta = \tan^{-1}(y/x)$: $\theta_x = \dfrac{-y}{x^2+y^2}$ and $\theta_y = \dfrac{x}{x^2+y^2} \Rightarrow \nabla\theta = \vec{F}$

(b) Because $\theta(x,y) = \tan^{-1}(y/x)$ is (well-) defined in the right-half plane $x > 0$ and $\vec{F} = \nabla\theta$, from the F. T. L. I.'s



$$\int_C \vec{F} \cdot d\vec{r} = \theta(x_2,y_2) - \theta(x_1,y_2) = \theta_2 - \theta_1$$

(c)



$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^{\pi} \frac{(-\sin(\theta))(-\sin(\theta)) + \cos(\theta)\cos(\theta)}{\cos^2(\theta) + \sin^2(\theta)}\, d\theta = \pi$$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^{-\pi} \frac{\sin^2(\theta) + \cos^2(\theta)}{\sin^2(\theta) + \cos^2(\theta)}\, d\theta = -\pi$$

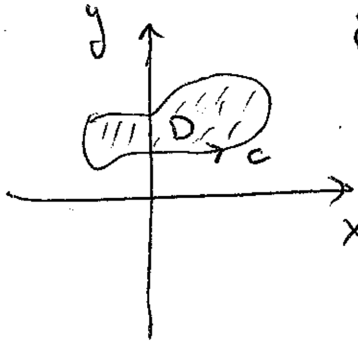
(d) $\vec{F}$ is not conservative over its entire domain since $\int_{C_1} \neq \int_{C_2}$. However, $\vec{F}$ is conservative over the right half-plane $x > 0$ as $\vec{F} = \nabla g$ on this region.

(e) $$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\Rightarrow \quad P_y = Q_x$$

(f) Let c be any simple closed curve that does not pass through or enclose the origin, and let D be the region given by the interior of c. Then, from Green's theorem:



$$\oint_c \vec{F}\cdot d\vec{r} = \iint_D \text{curl}(\vec{F})\,dA = \iint_D \underbrace{(P_y - Q_x)}_{=0}\,dA$$

$$= 0.$$

As we've seen in class, and from (c) this is not true for a curve which encloses (or passes through) the origin. In fact, let c be a curve enclosing the origin, then $\oint_c \vec{F}\cdot d\vec{r} = 2\pi\,\nu(c)$, where $\nu(c)\in\mathbb{Z}$ is the number of times c winds around the origin (+ if c.c. and − in clockwise).

Exercise 3 (5 points).

There is a famous scene in the movie "A Beautiful Mind" (see http://www.youtube.com/watch?v=pYdjNeFh6zw) where a too-busy-to-be-bothered John Nash (Russell Crowe) writes the following question on the board:

Find a subset X of $\mathbb{R}^3$ with the property that if V is the set of vector fields $\vec{F}$ on $\mathbb{R}^3 \setminus X$ which satisfy curl$(\vec{F}) = \vec{0}$ and W is the set of vector fields $\vec{G}$ which are conservative, $\vec{G} = \nabla f$, then the space V/W is 8-dimensional.

First a little explanation, by saying "the space V/W is 8 dimensional" he means that there should be in total 8 vector fields $\vec{F}_i$ which are not conservative (i.e., not the gradient of any functions) and which have vanishing curl outside of the set X. In addition, you cannot write any one of these vector fields as a linear combination of the other vector fields. Now, contrary to what Nash says (that it will take most students their entire life to solve) you can actually solve this question in a matter of hours, if that. However, since we've yet to enter the realm of vector fields on $\mathbb{R}^3$, we'll reword this problem for $\mathbb{R}^2$. Namely, find a subset X of $\mathbb{R}^2$ with the property that if V is the set of vector fields $\vec{F}$ on $\mathbb{R}^2 \setminus X$ which satisfy curl$(\vec{F}) = 0$ and W is the set of vector fields $\vec{G}$ which are conservative, $\vec{G} = \nabla f$, then the space V/W is 1-dimensional.

Let $X = \{(0,0)\}$, then $\vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ satisfies curl$(\vec{F}) = 0$ on $\mathbb{R}^2 \setminus X$, but $\vec{F}$ is not conservative.

Hence V/W contains at least the vector field $\vec{F} = \frac{1}{x^2+y^2}(-y\hat{i} + x\hat{j})$, which was deemed sufficient to solve the problem.