Math 53 - Multivariable Calculus

Homework # 3

July 12th
Due: July 17th, 2012

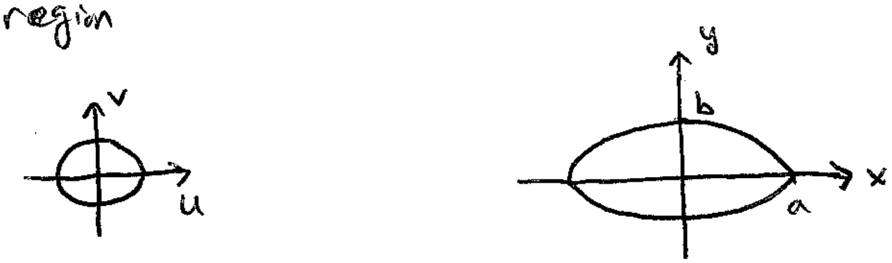
Solns

Exercise 1 (5 points).

(a) Describe and sketch the image of the disk of radius 1, $D = u^2 + v^2 \leq 1$, under the transformation $x = au$, $y = bv$, where $a, b \in \mathbb{R}$ and $a \neq 0$, $b \neq 0$.

Substituting $u = \frac{x}{a}$, $v = \frac{y}{b}$ into $u^2 + v^2 \leq 1$ gives $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$.

Thus, the image of the unit disk (w/ boundary) maps to the elliptical region



(b) Use the ideas from part (a) to find the area enclosed by the ellipse $(2x + 5y - 3)^2 + (3x - 7y + 8)^2 = 1$.

Let $u = 2x + 5y - 3$ and $v = 3x - 7y + 8$, then in the uv-plane the region of integration is a disk of radius 1, $u^2 + v^2 \leq 1$. Further,

$$J(x,y) = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 5 & -7 \end{vmatrix} = -29. \text{ Thus,}$$

$$\text{Area} = \iint_{\substack{\text{Disk} \\ r=1}} \frac{1}{|J(x,y)|}\, du\, dv = \frac{1}{29}\,\text{Area(Disk)} = \frac{\pi}{29}$$

Exercise 2 (5 points).

Evaluate $\displaystyle\int_{\mathbb{R}} e^{-ax^2}\,dx$, where a is any positive real number. (Hint: Let $I = \int_{\mathbb{R}} e^{-ax^2}\,dx$ then write I^2 as a double integral.)

If $I = \displaystyle\int_{\mathbb{R}} e^{-ax^2}\,dx$ then $I^2 = \displaystyle\iint_{\mathbb{R}^2} e^{-a(x^2+y^2)}\,dx\,dy$. Now convert to polar coordinates to give $I^2 = \displaystyle\int_0^{2\pi}\int_0^{\infty} e^{-ar^2}\,r\,dr\,d\theta$, which we can compute:

$$\int_0^{2\pi}\underbrace{\int_0^{\infty} e^{-ar^2}\,r\,dr}\,d\theta = -\frac{1}{2}\int_0^{2\pi}\left(\frac{-1}{a}\right)d\theta = \frac{\pi}{a}$$

let $u = -r^2 \Rightarrow du = -2r\,dr$

$$\int e^{au}\,du = e^{au} = \frac{e^{-r^2}}{a}\Big|_0^{\infty} = \frac{0-1}{a}$$

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Hence $I = \sqrt{\dfrac{\pi}{a}}$

Exercise 3 (5 points).

Show that a constant force field does zero work on a particle that winds uniformly w times around the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.

Let $\vec{r}(t) = \langle a\cos(t), b\sin(t)\rangle$ $\quad 0 \le t \le 2\pi w$ ($w \in \mathbb{Z}$) and let $\vec{F} = \langle m, n\rangle$, $m,n \in \mathbb{R}$.

Then $\text{Work} = \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi w} \langle m,n\rangle \cdot \langle -a\sin(t), b\cos(t)\rangle \, dt$

$$= \int_0^{2\pi w} \left(-ma\sin(t) + nb\cos(t)\right) dt$$

$$= \left[ma\cos(t) + nb\sin(t)\right]\Big|_0^{2\pi w}$$

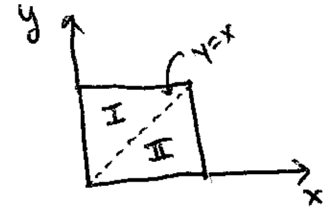
$$= (am + 0) - (am + 0)$$

$$= 0$$

Exercise 4 (5 points).

Compute $\int_0^1 \int_0^1 e^{\max(x^2,y^2)}\, dydx$ where $\max(x^2, y^2)$ means the larger of the numbers x^2 and y^2.

We first chop up our region:



Note, in region I $y^2 > x^2$ while in region II $x^2 > y^2$. Further

$\Rightarrow \int_0^1\int_0^1 e^{\max(x^2,y^2)}\, dydx$ is equal to the sum $\iint_{I} + \iint_{II}$. So:

$$\iint_{I} e^{\max(x^2,y^2)}\, dA = \int_0^1\int_0^y e^{y^2}\, dx\, dy = \int_0^1 y e^{y^2}\, dy = \frac{e-1}{2}$$

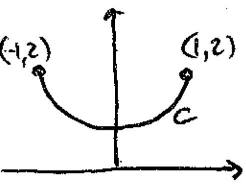
$$\iint_{II} e^{\max(x^2,y^2)}\, dA = \int_0^1\int_0^x e^{x^2}\, dy\, dx = \int_0^1 x e^{x^2}\, dx = \frac{e-1}{2}$$

Thus, $\int_0^1\int_0^1 e^{\max(x^2,y^2)}\, dA = \frac{e-1}{2} + \frac{e-1}{2} = e-1$

Exercise 5 (5 points).

Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$ and C is the parabola $y = 1 + x^2$ from $(-1,2)$ to $(1,2)$.

Note, C is given by $\vec{r}(t) = \langle t, 1+t^2 \rangle \quad -1 \le t \le 1.$



Thus, $\vec{F}(\vec{r}(t)) = \left\langle \frac{t}{\sqrt{t^2+(1+t^2)^2}}, \frac{1+t^2}{\sqrt{t^2+(1+t^2)^2}} \right\rangle$ and

$\vec{r}\,'(t) = \langle 1, 2t \rangle.$ So,

$$\int_C \vec{F}\cdot d\vec{r} = \int_{-1}^{1} \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt$$

$$= \int_{-1}^{1} \left[\frac{t}{\sqrt{t^2+(t^2+1)^2}} + \frac{2t(1+t^2)}{\sqrt{t^2+(1+t^2)^2}} \right] dt$$

$$= \int_{-1}^{1} \underbrace{t}_{\text{odd } f^n} \cdot \underbrace{\frac{(3+2t^2)}{\sqrt{t^2+(1+t^2)^2}}}_{\text{even } f^n}\, dt$$

and product is an odd f^n

$$= \int_{-1}^{1} (\text{odd } f^n)\, dt = 0$$