Exercise 1. (a) Show that if a particle moves with constant speed then the velocity and acceleration vectors are orthogonal. (b) If a particle with mass \( m \) moves with position vector \( \vec{r}(t) \) then its angular momentum is defined as \( \vec{L}(t) := m\vec{r}(t) \times \vec{v}(t) \) and its torque is defined as \( \vec{T}(t) := m\vec{r}(t) \times \vec{a}(t) \). Show that \( \frac{d}{dt}\vec{L}(t) = \vec{T}(t) \) and deduce that if \( \vec{T}(t) = 0 \) for all \( t \), then \( \vec{L}(t) \) is constant. This is the law of conservation of angular momentum.

(a) If \( |\vec{v}(t)| \) is constant then so is \( |\vec{v}(t)|^2 = \vec{v} \cdot \vec{v} \). Hence, \( \frac{d}{dt}(\vec{v} \cdot \vec{v}) = 0 \)

\[ \Rightarrow \vec{v}' \cdot \vec{v} + \vec{v} \cdot \vec{v}' = 2\vec{v} \cdot \vec{v} = 0 \Rightarrow \vec{a} \cdot \vec{v} = 0 \Rightarrow \vec{a} \perp \vec{v}. \]

(b) \( \frac{d}{dt} \vec{L}(t) = \frac{d}{dt}(m\vec{r}(t) \times \vec{v}(t)) = m \frac{d}{dt}(\vec{r} \times \vec{v}) = m(\vec{r} \times \vec{v}') + m(\vec{r}' \times \vec{v}) = m(\vec{r} \times \vec{a} \)
Exercise 2. Suppose you need to know the equation of a tangent plane to a surface $S$ at the point $p = (2, 1, 3)$. You don’t have an equation for $S$ but you do know that the curves

$$\vec{r}_1(t) = (2 + 3t, 1 - t^2, 3 - 4t + t^2), \quad \vec{r}_2(u) = (1 + u^2, 2u^3 - 1, 2u + 1),$$

both lie on $S$. Find an equation of the tangent plane to $S$ at $p$.

Since $\vec{r}_1(t)$ and $\vec{r}_2(u)$ lie on $S$, we can get tangent vectors at $p$ by $\vec{V}_1(0) = \vec{r}_1'(0) = \langle 3, 0, -4 \rangle$ and $\vec{V}_2(0) = \vec{r}_2'(0) = \langle 2, 6, 2 \rangle$. So,

$$\vec{N} = \langle 3, 0, -4 \rangle \times \langle 2, 6, 2 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & -4 \\ 2 & 6 & 2 \end{vmatrix} = \langle 24, -12, 18 \rangle.$$ So, since $\vec{P}_0 = \langle x - 2, y - 1, z - 3 \rangle$, the eqn of the tangent plane is

$$\langle 24, -12, 18 \rangle \cdot \langle x - 2, y - 1, z - 3 \rangle = 0$$

$$24x - 12y + 18z = 90$$
Exercise 3. Find the absolute maximum and minimum values of the function given by \( f(x, y) = 4x + 6y - x^2 - y^2 \) on the set \( D = \{(x, y) \mid 0 \leq x \leq 4, \ 0 \leq y \leq 5\} \).

Since \( D \) is compact and \( f(x,y) \) is continuous, we know that both an absolute min and absolute max exist. Let's first find the critical point(s) in the interior of \( D \). Following the general procedure, \( f_x = 0 \) and \( f_y = 0 \) \( \Rightarrow \) critical point is \( (2,3) \) and \( f(2,3) = 4(2) + 6(3) - (2)^2 - (3)^2 = 13 \). Now we must evaluate \( f \) on the boundary of \( D \), \( \partial D \), to see whether \( (2,3) \) is an absolute min or absolute max or neither, and find the other critical point(s).

The boundary of \( D \) is \( \partial D = L_1 \cup L_2 \cup L_3 \cup L_4 \). Along \( L_1 : y = 0 \) \( \Rightarrow f(x,0) = -(x-2)^2 + 4 \), \( 0 \leq x \leq 4 \), which has max @ \( (2,0) \) \( \Rightarrow \) \( f(2,0) = 4 \) and min @ \( (0,0) \) and \( (4,0) \) \( \Rightarrow \) \( f(0,0) = f(4,0) = 0 \). Similarly, on \( L_2 \) \( f \) has max @ \( (4,3) \) \( \Rightarrow \) \( f(4,3) = 9 \) and min @ \( (4,0) \) \( \Rightarrow \) \( f(4,0) = 0 \). On \( L_3 \) \( f \) has max @ \( (2,5) \) \( \Rightarrow \) \( f(2,5) = 9 \) and min @ \( (0,5) \) and \( (4,5) \) \( \Rightarrow \) \( f(4,5) = 5 \). Finally, along \( L_4 \) \( f \) has max @ \( (0,3) \) \( \Rightarrow \) \( f(0,3) = 9 \) and min @ \( (0,0) \) \( \Rightarrow \) \( f(0,0) = 0 \).

Thus, the absolute maximum value obtained by \( f \) on \( D \) occurs @ \( (2,3) \), with value \( f(2,3) = 13 \), and the absolute min occurs @ \( (0,0) \) and \( (4,0) \), with \( f(0,0) = f(4,0) = 0 \).
Exercise 4. Find three positive numbers whose sum is 12 and the sum of whose squares is as small as possible.

To do so, we want to minimize \( f(x, y) = x^2 + y^2 + (12-x-y)^2 \). Now,

\[
0 = f_x = 2x + 2(12-x-y) \iff x = 12 + x + y = 0, \quad \text{and also}
\]

\[
0 = f_y = 2y + 2(12-x-y) \iff y = 12 + x + y = 0; \quad \text{i.e.,} \quad x = 4, \quad y = 4 \implies z = 4.
\]

So, the point \((4, 4, 4)\) is a critical pt. and since \( f_{xx} = 2, \ f_{yy} = 2, \ f_{xy} = 2 \)

we have \( (f_{xx})^2 (f_{yy})^2 - (f_{xy})^2 = 12 > 0 \) \( \implies \) \((4, 4, 4)\) is a minimum.

Hence, we conclude that the three positive numbers which satisfy the constraints are \( 4, 4, 4 \).