Math 53 - Multivariable Calculus

Sol'ns

Homework # 2

June 28th
Due: July 3rd, 2012

Exercise 1. (*a*) *Show that if a particle moves with constant speed then the velocity and acceleration vectors are orthogonal.*
(*b*) *If a particle with mass m moves with position vector $\vec{r}(t)$ then its angular momentum is defined as $\vec{l}(t) := m\vec{r}(t) \times \vec{v}(t)$ and its torque is defined as $\vec{\tau}(t) := m\vec{r}(t) \times \vec{a}(t)$. Show that $\frac{d}{dt}\vec{l}(t) = \vec{\tau}(t)$ and deduce that if $\vec{\tau}(t) = 0$ for all t, then $\vec{l}(t)$ is constant. This is the law of conservation of angular momentum.*

(a) If $|\vec{v}(t)|$ is constant then so is $|\vec{v}(t)|^2 = \vec{v}\cdot\vec{v}$. Hence, $\frac{d}{dt}(\vec{v}\cdot\vec{v}) = 0$

$\Leftrightarrow \vec{v}'\cdot\vec{v} + \vec{v}\cdot\vec{v}' = 2\vec{v}'\cdot\vec{v} = 0 \Rightarrow \vec{a}\cdot\vec{v} = 0 \Rightarrow \vec{a} \perp \vec{v}$.

(b) $\frac{d}{dt}\vec{l}(t) = \frac{d}{dt}(m\vec{r}(t)\times\vec{v}(t)) = m\frac{d}{dt}(\vec{r}\times\vec{v}) = m(\vec{r}'\times\vec{v} + \vec{r}\times\vec{v}') = m(\underbrace{\cancel{\vec{v}\times\vec{v}}}_{0} + \vec{r}\times\vec{a})$

$= m\vec{r}\times\vec{a}$.

Exercise 2. *Suppose you need to know the equation of a tangent plane to a surface S at the point $p = (2,1,3)$. You don't have an equation for S but you do know that the curves*

$$\vec{r}_1(t) = \langle 2+3t, 1-t^2, 3-4t+t^2\rangle, \qquad \vec{r}_2(u) = \langle 1+u^2, 2u^3-1, 2u+1\rangle,$$

both lie on S. Find an equation of the tangent plane to S at p.

Since $\vec{r}_1(t)$ and $\vec{r}_2(u)$ lie on S, we can get tangent vectors at p by $\vec{v}_1(0) = \vec{r}_1{}'(0) = \langle 3,0,-4\rangle$ and $\vec{v}_2(0) = \vec{r}_2{}'(1) = \langle 2,6,2\rangle$. So,

$$\vec{N} = \langle 3,0,-4\rangle \times \langle 2,6,2\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & -4 \\ 2 & 6 & 2 \end{vmatrix} = \langle 24, -12, 18\rangle.$$

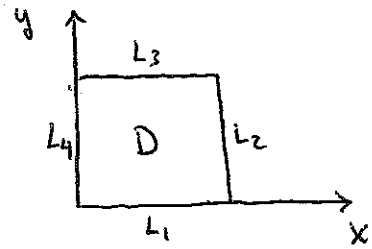
So, since $\vec{P_0P} = \langle x-2, y-1, z-3\rangle$, the eq$^{\underline{n}}$ of the tangent plane is

$$\langle 24,-12,18\rangle \cdot \langle x-2, y-1, z-3\rangle = 0$$

$$\Updownarrow$$

$$24x - 12y + 18z = 90$$

Exercise 3. *Find the absolute maximum and minimum values of the function given by* $f(x,y) = 4x + 6y - x^2 - y^2$ *on the set* $D = \{(x,y) \mid 0 \le x \le 4,\ 0 \le y \le 5\}$.



Since D is compact and $f(x,y)$ is continuous, we know that both an absolute min and absolute max exist. Let's first find the critical point(s) in the interior of D, $\mathring{D}$. Following the general procedure, $f_x = 4 - 2x = 0$ and $f_y = 6 - 2y \Rightarrow$ critical point is $(2,3)$ and $f(2,3) = 4(2) + 6(3) - (2)^2 - (3)^2 = 13$. Now we must evaluate f on the boundary of D, ∂D, to see whether $(2,3)$ is an absolute min or absolute max or neither, and find the other critical point(s).

The boundary of D is $\partial D = L_1 \cup L_2 \cup L_3 \cup L_4$. Along L_1: $y = 0 \Rightarrow$ $f(x,0) = -(x-2)^2 + 4$, $0 \le x \le 4$, which has max @ $(2,0)$ w/ $f(2,0) = 4$ and min @ $(0,0)$ and $(4,0)$ w/ $f(0,0) = f(4,0) = 0$. Similarly, on L_2 f has max @ $(4,3)$ w/ $f(4,3) = 9$ and min @ $(4,0)$ w/ $f(4,0) = 0$. On L_3 f has max @ $(2,5)$ w/ $f(2,5) = 9$ and min @ $(0,5)$ and $(4,5)$ w/ $f(4,5) = 5$. Finally, along L_4 f has max @ $(0,3)$ w/ $f(0,3) = 9$ and min @ $(0,0)$ w/ $f(0,0) = 0$.

Thus, the absolute maximum value obtained by f on D occurs @ $(2,3)$, with value $f(2,3) = 13$, and the absolute min occurs @ $(0,0)$ and $(4,0)$, with $f(0,0) = f(4,0) = 0$.

Exercise 4. *Find three positive numbers whose sum is 12 and the sum of whose squares is as small as possible.*

To do so, we want to minimize $f(x,y) = x^2 + y^2 + (12-x-y)^2$. Now,

$0 = f_x = 2x - 2(12-x-y) \iff x - 12 + x + y = 0$, and also

$0 = f_y = 2y - 2(12-x-y) \iff y - 12 + x + y = 0$; i.e., $x = 4$, $y = 4 \Rightarrow z = 4$.

So, the point $(4,4,4)$ is a critical pt. and since $f_{xx} = 4$, $f_{yy} = 4$, $f_{xy} = 2$

we have $(f_{xx})(f_{yy}) - (f_{xy})^2 = 12 > 0 \Rightarrow (4,4,4)$ is a minimum.

Hence, we conclude that the three positive numbers which satisfy the constraints are $4, 4, 4$.