Exercise 1 (Stewart 12.3 # 49). Use a scalar projection to show that the distance from a point \( P_1 = (x_1, y_1) \in \mathbb{R}^2 \) to the line \( ax + by + c = 0 \) is

\[
\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.
\]

Use this formula to find the distance from the point \((-2, 3)\) to the line \(3x - 4y + 5 = 0\).

To begin, note that \( \vec{N} = (a, b) \) is perpendicular to the line. Indeed, let \((a_1, b_1)\) and \((a_2, b_2)\) be two points on the line. Then

\[
\vec{N} \cdot (a_2 - a_1, b_2 - b_1) = a(a_2 - a_1) + b(b_2 - b_1) = a_2a + b_2b - (a_1a + b_1b) = -c + c = 0.
\]

Next, let \( P_2 = (x_2, y_2) \) lie on the line. Then the distance from \( P_1 \) to the line is the absolute value of the scalar projection of \( \overrightarrow{P_1P_2} \) onto \( \vec{N} \). So, we have

\[
d = \operatorname{comp}_{\vec{N}} \overrightarrow{P_1P_2} = \frac{|\vec{N} \cdot (x_2 - x_1, y_2 - y_1)|}{|\vec{N}|} = \frac{|ax_2 - ax_1 + by_2 - by_1|}{\sqrt{a^2 + b^2}} = \frac{|ax_2 + by_2 - ax_1 - by_1|}{\sqrt{a^2 + b^2}} = \frac{|-c - ax_1 - by_1|}{\sqrt{a^2 + b^2}}.
\]

From this formula, it follows that the distance from \((-2, 3)\) to \(3x - 4y + 5 = 0\) is

\[
\frac{|3(-2) - 4(3) + 5|}{\sqrt{3^2 + 4^2}} = \frac{13}{5}.
\]
Exercise 2 (Stewart 12.3 # 57). Use that $\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos(\theta)$ to prove the Cauchy-Schwarz inequality:

$$|\vec{A} \cdot \vec{B}| \leq |\vec{A}| |\vec{B}|.$$ 

$$|\vec{A} \cdot \vec{B}| = |\vec{A}||\vec{B}| \cos(\theta) = |\vec{A}| |\vec{B}| |\cos(\theta)| \leq |\vec{A}| |\vec{B}|$$

since $|\cos(\theta)| \leq 1.$
Exercise 3 (Stewart 12.4 # 44). (a) Let $P$ be a point not on the plane that passes through the points $Q$, $R$, and $S$. Show that the distance $d$ from $P$ to the plane is

$$d = \frac{|(\vec{A} \times \vec{B}) \cdot \vec{C}|}{|\vec{A} \times \vec{B}|},$$

where $\vec{A} = \overrightarrow{QR}$, $\vec{B} = \overrightarrow{QS}$, and $\vec{C} = \overrightarrow{QP}$. (b) Use the formula from before to find the distance from the point $P(2,1,4)$ to the plane through the points $Q(1,0,0)$, $R(0,2,0)$, and $S(0,0,3)$.

(a) Up to sign, the distance $d$ from $P$ to the plane is the component of $\vec{C} = \overrightarrow{QP}$ along the direction of $\vec{A} \times \vec{B}$. Hence,

$$d = \frac{|\vec{C} \cdot (\vec{A} \times \vec{B})|}{|\vec{A} \times \vec{B}|}$$

(b) $\vec{A} = \langle -1,2,0 \rangle$, $\vec{B} = \langle -1,0,3 \rangle$, $\vec{C} = \langle 1,1,4 \rangle$, and $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = \langle 6,3,0 \rangle$.

$\vec{A} \times \vec{B} \cdot \vec{C} = 17 \Rightarrow d = \frac{|17|}{\sqrt{36 + 9 + 9}} = \frac{17}{7}$. 
Exercise 4 (The Geometry of a Tetrahedron, Stewart page 794). (Part 1) Let $\vec{V}_1$, $\vec{V}_2$, $\vec{V}_3$, and $\vec{V}_4$ be vectors with lengths equal to the areas of the faces opposite the vertices $P$, $Q$, $R$, and $S$, respectively, and directions perpendicular to the respective faces an pointing outward. Show that

$$\sum_{i=1}^{4} \vec{V}_i = \vec{0}.$$ 

(Part 2) Suppose the tetrahedron in the figure (see Stewart page 794) has a trirectangular vertex $S$. Let $A$, $B$, and $C$ be the areas of the three faces that meet at $S$, and let $D$ be the area of the opposite face $PQR$. Using the results of part 1, or otherwise, show that


(A) The vector coming out of the face opposite $P$ (i.e., the bottom face) is $\vec{V}_1 = \frac{1}{2} \vec{a} \times \vec{b}$. Indeed $|\vec{V}_1| = A$ (4a) and points downward. For the face opposite $Q$, we have $\vec{V}_2 = \frac{1}{2} \vec{a} \times \vec{c}$. For the face opposite $R$ we have $\vec{V}_3 = \frac{1}{2} \vec{b} \times \vec{a}$. Finally, for the face opposite $S$, we have $\vec{V}_4 = \frac{1}{2} \vec{PQ} \times \vec{PR} = \frac{1}{2} (\vec{b} - \vec{a}) \times (\vec{c} - \vec{a}) = \frac{1}{2} (\vec{b} \times \vec{a} - \vec{a} \times \vec{c} + \vec{0}) = -\vec{V}_1 - \vec{V}_2 - \vec{V}_3$. So, $\sum_{i=1}^{4} \vec{V}_i = \vec{0}.$

(b) We can set up our coordinate system so that $S$ is at the origin, $SQ$ is the $x$-axis, $SR$ is the $y$-axis and $SP$ is the $z$-axis. Now, the face opposite $P$ is in the $xy$-plane and has area $A \Rightarrow \vec{V}_1 = \langle 0, 0, -A \rangle$. Similarly, $\vec{V}_2 = \langle -B, 0, 0 \rangle$, $\vec{V}_3 = \langle 0, -C, 0 \rangle \Rightarrow \vec{V}_4 = \langle B, C, A \rangle$. So, if the area of the fourth face is $D = |\vec{V}_4| \Rightarrow D = |\vec{V}_4| = \sqrt{B^2 + C^2 + A^2}$ or $D^2 = A^2 + B^2 + C^2$. 

Exercise 5 (Stewart 12.5 #13, #52). (a) Is the line through \((-4, -6, 1)\) and \((-2, 0, -3)\) perpendicular to the line through \((-3, 2, 0)\) and \((5, 1, 4)\)?
(b) Determine whether the planes
\[2x - 3y + 4z = 5, \quad x + 6y + 4z = 3,\]
are parallel, perpendicular, or neither. If neither, find the angle between them.

(a) Let \(L_1\) be the line through \((-4, -6, 1)\) and \((-2, 0, -3)\) and \(L_2\) the other line. Then \(L_1\) has directional vector \(\vec{v}_1 = \langle 2, 6, -4 \rangle\) and \(\vec{v}_2 = \langle 8, -1, 4 \rangle\). So, \(\vec{v}_1 \cdot \vec{v}_2 = 16 - 6 - 16 \neq 0 \Rightarrow \) \(L_1\) is not perpendicular to \(L_2\).

(b) Here we have \(\vec{N}_1 = \langle 2, -3, 4 \rangle\) and \(\vec{N}_2 = \langle 1, 6, 4 \rangle\) for the normal vectors. Since \(\vec{N}_1 \cdot \vec{N}_2 = 2 - 18 + 16 = 0\), the planes are \(\perp\).