Math [illegible] - Multivariable Calculus

Homework # 1

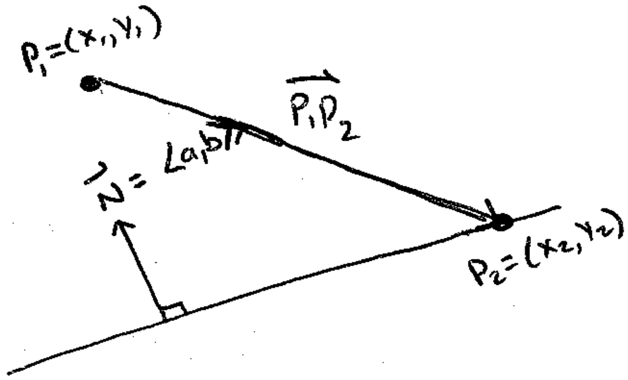
Solns

June 21st
Due: June 26th, 2012

Exercise 1 (Stewart 12.3 # 49). *Use a scalar projection to show that the distance from a point $P_1 = (x_1, y_1) \in \mathbb{R}^2$ to the line $ax + by + c = 0$ is*

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$

Use this formula to find the distance from the point $(-2, 3)$ to the line $3x - 4y + 5 = 0$.



To begin, note that $\vec{N} = \langle a, b \rangle$ is $\perp$ to the line. Indeed let (a_1, b_1) and (a_2, b_2) be two points on the line. Then $\vec{N} \cdot \langle a_2 - a_1, b_2 - b_1 \rangle = a(a_2 - a_1) + b(b_2 - b_1)$ $= \underbrace{aa_2 + bb_2}_{=-c} - (\underbrace{aa_1 + bb_1}_{=-c}) = -c + c = 0.$

Next, let $P_2 = (x_2, y_2)$ lie on the line. Then the distance from P_1 to the line is the absolute value of the scalar projection of $\overrightarrow{P_1P_2}$ onto $\vec{N}$. So, we have $d = \text{comp}_{\hat{N}} \overrightarrow{P_1P_2} = \dfrac{|\vec{N} \cdot \langle x_2 - x_1, y_2 - y_1 \rangle|}{|\vec{N}|}$

$$= \frac{|ax_2 - ax_1 + by_2 - by_1|}{\sqrt{a^2+b^2}} = \frac{|ax_2 + by_2 - ax_1 - by_1|}{\sqrt{a^2+b^2}} = \frac{|-c - ax_1 - by_1|}{\sqrt{a^2+b^2}}$$

$= \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$. From this formula, it follows that the distance from $(-2, 3)$ to $3x - 4y + 5 = 0$ is $\dfrac{|(3)(-2) + (-4)(3) + 5|}{\sqrt{3^2 + 4^2}} = \dfrac{13}{5}$.

Exercise 2 (Stewart 12.3 # 57). *Use that $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|\cos(\theta)$ to prove the Cauchy-Schwarz inequality:*

$$|\vec{A} \cdot \vec{B}| \leq |\vec{A}||\vec{B}|.$$

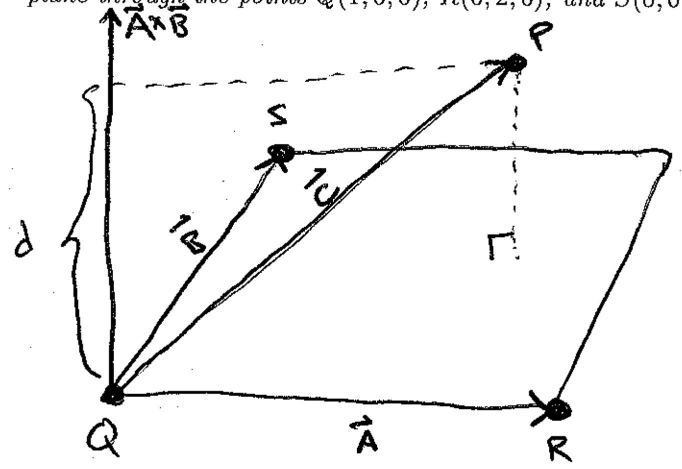
$$|\vec{A} \cdot \vec{B}| = \big|\, |\vec{A}||\vec{B}| \cos(\theta) \big| = |\vec{A}||\vec{B}| |\cos(\theta)| \leq |\vec{A}||\vec{B}|$$

since $|\cos(\theta)| \leq 1$.

Exercise 3 (Stewart 12.4 # 44). *(a) Let P be a point not on the plane that passes through the points Q, R, and S. Show that the distance d from P to the plane is*

$$d = \frac{|(\vec{A} \times \vec{B}) \cdot \vec{C}|}{|\vec{A} \times \vec{B}|},$$

where $\vec{A} = \vec{QR}$, $\vec{B} = \vec{QS}$, and $\vec{C} = \vec{QP}$. (b) Use the formula from before to find the distance from the point $P(2, 1, 4)$ to the plane through the points $Q(1, 0, 0)$, $R(0, 2, 0)$, and $S(0, 0, 3)$.



(a) Up to sign, the distance d from P to the plane is the component of $\vec{C} = \vec{QP}$ along the direction of $\vec{A} \times \vec{B}$. Hence,

$$d = \frac{|\vec{C} \cdot (\vec{A} \times \vec{B})|}{|\vec{A} \times \vec{B}|}$$

(b) $\vec{A} = \langle -1, 2, 0 \rangle$, $\vec{B} = \langle -1, 0, 3 \rangle$, $\vec{C} = \langle 1, 1, 4 \rangle$, and $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = \langle 6, 3, 2 \rangle$.

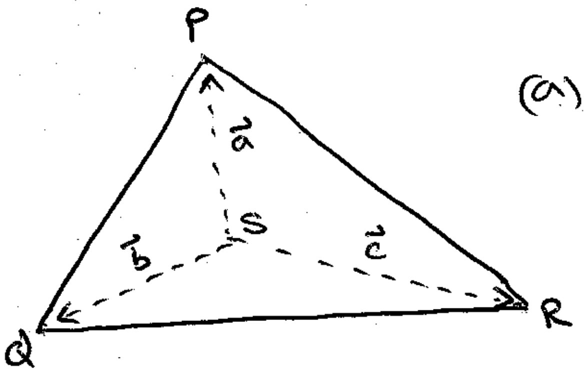
$\vec{A} \times \vec{B} \cdot \vec{C} = 17 \Rightarrow d = \dfrac{|17|}{\sqrt{36+9+4}} = \dfrac{17}{7}$.

Exercise 4 (The Geometry of a Tetrahedron, Stewart page 794). (*Part 1*) *Let $\vec{V}_1$, $\vec{V}_2$, $\vec{V}_3$, and $\vec{V}_4$ be vectors with lengths equal to the areas of the faces opposite the vertices P, Q, R, and S, respectively, and directions perpendicular to the respective faces an pointing outward. Show that*

$$\sum_{i=1}^{4} \vec{V}_i = \vec{0}.$$

(*Part 2*) *Suppose the tetrahedron in the figure (see Stewart page 794) has a trirectangular vertex S. Let A, B, and C be the areas of the three faces that meet at S, and let D be the area of the opposite face PQR. Using the results of part 1, or otherwise, show that*

$$D^2 = A^2 + B^2 + C^2.$$



(a) The vector coming out of the face opposite P (i.e., the bottom face) is $\vec{V}_1 = \frac{1}{2}\vec{c} \times \vec{b}$. Indeed $|\vec{V}_1| = A(\Delta)$ and points downward. For the face opposite Q, we have $\vec{V}_2 = \frac{1}{2}\vec{a} \times \vec{c}$. Face opposite R we have $\vec{V}_3 = \frac{1}{2}\vec{b} \times \vec{a}$.

Finally, for the face opposite S, we have $\vec{V}_4 = \frac{1}{2}\overrightarrow{PQ} \times \overrightarrow{PR} = \frac{1}{2}(\vec{b}-\vec{a}) \times (\vec{c}-\vec{a})$
$= \frac{1}{2}\left(-\vec{c}\times\vec{b} - \vec{b}\times\vec{a} - \vec{a}\times\vec{c} + \vec{0}\right) = -\vec{V}_1 - \vec{V}_2 - \vec{V}_3$. So, $\sum_{i=1}^{4} \vec{V}_i = 0$.

(b) We can set up our coordinate ~~sys~~ system so that S is at the origin, SQ is the x-axis, SR is the y-axis and SP is the z-axis. Now, the face opposite P is in the xy-plane and has area $= A \Rightarrow$ $\vec{V}_1 = \langle 0,0,-A \rangle$. Similarly, $\vec{V}_2 = \langle -B,0,0 \rangle$, $\vec{V}_3 = \langle 0,-C,0 \rangle \Rightarrow$ $\vec{V}_4 = \langle B,C,A \rangle$. So, if the area of the fourth face is $D = |\vec{V}_4|$ $\Rightarrow D = |\vec{V}_4| = \sqrt{B^2 + C^2 + A^2}$ or $D^2 = A^2 + B^2 + C^2$.

Exercise 5 (Stewart 12.5 #13, #52). (*a*) *Is the line through* $(-4,-6,1)$ *and* $(-2,0,-3)$ *perpendicular to the line through* $(-3,2,0)$ *and* $(5,1,4)$*?*

(*b*) *Determine whether the planes*

$$2x - 3y + 4z = 5, \qquad x + 6y + 4z = 3,$$

are parallel, perpendicular, or neither. If neither, find the angle between them.

(a) Let L_1 be the line through $(-4,-6,1)$ and $(-2,0,-3)$ and L_2 the other line. Then L_1 has directional vector $\vec{v}_1 = \langle 2,6,-4 \rangle$ and $\vec{v}_2 = \langle 8,-1,4 \rangle$. So, $\vec{v}_1 \cdot \vec{v}_2 = 16 - 6 - 16 \neq 0 \Rightarrow L_1$ is not perpendicular to L_2.

(b) Here we have $\vec{N}_1 = \langle 2,-3,4 \rangle$ and $\vec{N}_2 = \langle 1,6,4 \rangle$ for the normal vectors. Since $\vec{N}_1 \cdot \vec{N}_2 = 2 - 18 + 16 = 0$, the planes are $\perp$.