Math 53 - Multivariable Calculus

Midterm # 2

August 4th, 2011

Exercise 1.

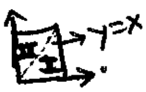
(*a*) (*5 points*) *Evalute the integral*

$$\int_0^1 \int_0^1 e^{max\{x^2,y^2\}} dydx,$$

where $max\{x^2, y^2\}$ *means the larger of the numbers* x^2 *and* y^2.

(*b*) (*5 points*) *Find the volume of the solid E that lies above the cone* $z = \sqrt{x^2+y^2}$ *and below the sphere* $x^2+y^2+z^2=1$.

(a) We're integrating over the unit square 

. Now, along the line $y=x$, $\max\{x^2,y^2\}=x^2=y^2$ so this is where we split our region of integration: 

In region I, $x^2>y^2 \Rightarrow \max\{x^2,y^2\}=x^2 \Rightarrow$ we're integrating $\int_0^1\int_0^x e^{x^2}dydx = \int_0^1 xe^{x^2}dx = \frac{1}{2}e-\frac{1}{2}$. Similarly, in region II $y^2>x^2 \Rightarrow \max\{x^2,y^2\}=y^2 \Rightarrow$ integral becomes $\int_0^1\int_0^y ye^{y^2}dxdy = \frac{1}{2}e-\frac{1}{2}$. So, $\int_0^1\int_0^1 e^{\max\{x^2,y^2\}}dydx = e-1$.

(b) In spherical coordinates $z=\sqrt{x^2+y^2}$ becomes $\cos(\varphi)=\sin(\varphi) \Leftrightarrow \varphi=\pi/4$ (since $\cos(\varphi)$ is $\pi/4$-phase $\sin(\varphi)$). Then

$$V=\int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^2\sin(\varphi)\,d\rho\, d\varphi\, d\theta = \left(\int_0^{2\pi}d\theta\right)\left(\int_0^{\pi/4}\sin(\varphi)\,d\varphi\right)\left(\int_0^1\rho^2\,d\rho\right)$$

$$=\frac{1}{3}\pi\left(2-\sqrt{2}\right).$$

Exercise 2. (5 *points*) *How many functions* $f : \mathbb{R}^2 \to \mathbb{R}$ *are there whose partial derivatives are* $f_x(x, y) = x + 4y$ *and* $f_y(x, y) = 3x - y$. *Prove your response.*

There are exactly 0 such fns. Indeed, suppose an f existed. Then $f_x = x+4y \Rightarrow f_{xy} = 4$ and $f_y = 3x - y \Rightarrow f_{yx} = 3$. Since f_{xy} and f_{yx} are continuous everywhere but $f_{xy} \neq f_{yx}$, by Clairaut's theorem, no such f exists; this f contradicts Clairaut's theorem.

Exercise 3. (10 *points*) *Show that a constant force field does zero work on a particle that moves once uniformly around the circle $x^2 + y^2 = 1$. Is this also ture for a force field $\vec{F}(\vec{x}) = k\vec{x}$, where $k \in \mathbb{R}$ and $\vec{x} = \langle x, y \rangle$.*

Let $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$, $0 \le t < 2\pi$, and let

$\vec{F} = \langle a, b \rangle$, $a, b \in \mathbb{R}$. Then

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (-a\sin(t) + b\cos(t))\,dt = a\cos(t) + b\sin(t) \Big|_0^{2\pi}$$

$$= a + 0 - a + 0 = 0.$$

This is also true for $\vec{F} = \langle kx, ky \rangle$. Indeed,

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (-k\sin(t)\cos(t) + k\sin(t)\cos(t))\,dt = \int_0^{2\pi} 0\,dt = 0$$

Exercise 4. (~~10 points~~ 15 points) *Compute the flux of $\vec{F} = \langle z^2 x, \frac{1}{3}y^3 + \tan(z), x^2 z + y^2 \rangle$ through the top half of the sphere S, $x^2 + y^2 + z^2 = 1$.*

We need to ~~[scribbled]~~ use the divergence theorem. However, S is not closed. So, we first integrate over S_1 and S_2, where S_1 is the disk $x^2+y^2 \le 1$ (oriented downward) and $S_2 = S \cup S_1$.

So, for S_1 $\hat{n} = -\hat{k} \Rightarrow \iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} (-x^2 z - y^2)\,ds = \iint_{S_1} -y^2\,ds$, since $z=0$ on disk. So, switching to polar coords., we have $-\int_0^{2\pi}\int_0^1 r^2 \sin^2(\theta)\, r\,dr\,d\theta = -\frac{1}{4}\pi$. Now, since S_2 is closed, we can use the divergence theorem. $\operatorname{div}(\vec{F}) = z^2 + y^2 + x^2$ and thus

$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \iiint \operatorname{div}(\vec{F})\,dv = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2 \cdot \rho^2 \sin(\phi)\,d\rho\,d\phi\,d\theta = \frac{2}{5}\pi.$

Finally $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_2} \vec{F}\cdot\hat{n}\,ds - \iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \frac{13}{20}\pi.$

Exercise 5.

(*a*) (5 *points*) *Show that the vector field*

$$\vec{F} = \langle e^x yz, e^x z + 2yz, e^x y + y^2 + 1 \rangle$$

is conservative.

(*b*) (10 *points*) *Find a potential for* $\vec{F}$.

(*c*) (5 *points*) *Prove whether the vector field* $\vec{G} = \langle y, x, y \rangle$ *is conservative or not. If so, find a potential for* $\vec{G}$.

(a) Since $\vec{F}$ def$^{\underline{nd}}$ and diff$^{\underline{ble}}$, just need to check that $M_y = N_x$, $M_z = P_x$, $N_z = P_y$, where $M = e^x yz$, $N = e^x z + 2yz$, $P = e^x y + y^2 + 1$.

So, $M_y = e^x z = N_x$, $M_z = e^x y = P_x$, $N_z = e^x + 2y = P_y$.

(b) We have

$$f_x = e^x yz$$
$$f_y = e^x z + 2yz$$
$$f_z = e^x y + y^2 + 1$$

Integrating f_x gives $f = e^x yz + g(y,z)$. Differentiating w.r.t. y and comparing with the above gives, $f_y = e^x z + g_y \overset{!}{=} e^x z + 2yz \Rightarrow g_y = 2yz$ $\Rightarrow g = y^2 z + h(z)$. Similarly, $g_z = y^2 + h' \overset{!}{=} + y^2 + 1 \Rightarrow h' = 1$ $\Rightarrow h = z + C$. Thus, $f = e^x yz + y^2 z + z + C$.

(c) The field is not conservative: $N_z = 0$ while $P_y = 1$.

Exercise 6. (10 *points*) *Suppose a particle moves along line segments from the origin to the points* $(1,0,0)$, $(1,2,1)$, $(0,2,1)$, *and back to the origin under the influence of the force field* $\vec{F} = \langle z^2, 2xy, 4y^2\rangle$. *Find the work done.*

It's easier to use Stokes' thm than to compute $\oint_C \vec{F}\cdot d\vec{r}$ directly. So, let S be the planar region enclosed by the path of the particle; i.e., S is the portion of the plane $z = \frac{1}{2}y$, $0 \le x \le 1$, $0 \le y \le 2$, with upward orientation. Further, $\text{curl}(\vec{F}) = \langle 8y, 2z, 2y\rangle$, and thus

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}(\vec{F})\cdot \hat{n}\, dS = \iint_D \left(-8y(0) - 2z\left(\tfrac{1}{2}\right) + 2y\right) dA$$

$$= \int_0^1 \int_0^2 \left(2y - \tfrac{1}{2}y\right) dy\, dx = 3.$$

15 points

Exercise 7. (~~10 points~~) *Let M be the solid cylinder $x^2 + y^2 \leq 1$, $0 \leq z \leq 1$ and let $\vec{F} = \langle xy, yz, zx \rangle$. Verify that the divergence theorem is true for $\vec{F}$ on the region M.*

$$\operatorname{div}(\vec{F}) = x + y + z.$$

$$\iiint_E \operatorname{div}(\vec{F})\, dV = \int_0^{2\pi} \int_0^1 \int_0^1 (r\cos(\theta) + r\sin(\theta) + z)\, r\, dz\, dr\, d\theta$$

$$= \pi/2.$$

Next, let S_1 be the top of the cylinder, S_2 the bottom, and S_3 the vertical edge. Then, on S_1, $z=1$ $\hat{n} = \hat{k}$ and $\vec{F} = \langle xy, y, x \rangle$

$$\Rightarrow \iint_{S_1} \vec{F} \cdot \hat{n}\, dS = \iint_{S_1} x\, dS = \int_0^{2\pi}\int_0^1 r\cos(\theta)\, r\, dr\, d\theta = 0$$

On S_2, $z = 0$ $\hat{n} = -k$ $\vec{F} = \langle xy, 0, 0 \rangle \Rightarrow \iint_{S_1} \vec{F} \cdot \hat{n}\, dS = \iint_{S_1} 0\, dS = 0.$

S_3 is given by $\vec{r}(\theta, z) = \langle \cos(\theta), \sin(\theta), z \rangle$ $0 \leq \theta \leq 2\pi$, $0 \leq z \leq 1$.

Now, $\hat{n} = \langle \cos(\theta), \sin(\theta), 0 \rangle \Rightarrow \iint_{S_3} \vec{F} \cdot \hat{n}\, dS = \int_0^{2\pi} \int_0^1 (\cos^2(\theta)\sin(\theta) + z\sin^2(\theta))\, dz\, d\theta$

$$= \pi/2.$$

15 points

Exercise 8. (~~X points~~) *Let D be the upper hemisphere of the unit sphere $x^2+y^2+z^2=1$. Use Stokes' theorem to evaluate*

$$\int\int_D (x^3e^y\hat{\imath} - 3x^2e^y\hat{\jmath})\cdot\hat{n}\,dS,$$

where $\hat{n}$ is the upward-pointing normal vector.

If we wish to use Stokes' theorem, we must express $x^3e^y\hat{\imath} - 3x^2e^y\hat{\jmath}$ as the curl of some $\vec{F}$. The formula for the curl is: Let $\vec{F} = \langle P, Q, R\rangle$

$$\operatorname{curl}(\vec{F}) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\imath} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

So, we need $\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = x^3e^y$, $\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = -3x^2e^y$, $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$. So guessing $P = Q = 0$ we see that $R = x^3e^y$, indeed $\operatorname{curl}(x^3e^y\hat{k}) = x^3e^y\hat{\imath} - 3x^2e^y\hat{\jmath}$. Hence, by Stokes', $\iint_D (x^3e^y\hat{\imath} - 3x^2e^y\hat{\jmath})\cdot\hat{n}\,dS = \oint_C x^3e^y\,dz$, where $C = \partial S$.

Since S upper hemisphere of the unit sphere, C is just the unit circle on the xy-plane, which is ~~oriented~~ oriented counter-clockwise. Then $\oint_C x^3e^y\,dz = 0$ since z is not changing over $C \Rightarrow \iint_D (x^3e^y\hat{\imath} - 3x^2e^y\hat{\jmath})\cdot\hat{n}\,dS = 0$.