# Math 53 - Multivariable Calculus

Take Home Assignment # 5

August 1st, 2011

**Exercise 1.** *What is the volume enclosed by the ellipsoid*

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1 \ ?$$

**Exercise 2.** *Prove that*

$$\int_0^1 \int_0^1 \int_0^1 \frac{1}{1+xyz}\, dxdydz = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3},$$

*and then use this to evaluate the triple integral correct to two decimal places.*

**Exercise 3.** *Let* $\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ *denote the unit sphere in* $\mathbb{R}^3$. *Evaluate the following surface integral over* $\mathcal{S}$*:*

$$\int\int_{\mathcal{S}} \left(x^2 + y + z\right) dA.$$

**Exercise 4.** *Let* $\mathcal{B} = \left\{ \vec{\boldsymbol{r}} \in \mathbb{R}^3 \mid \|\vec{\boldsymbol{r}}\| \leq 1 \right\}$ *denote the unit ball in* $\mathbb{R}^3$. *Let* $\vec{\boldsymbol{J}} = \langle J_1, J_2, J_3 \rangle$ *be a smooth vector field (i.e., its components are infinitely differentiable) on* $\mathbb{R}^3$ *that vanishes outside of* $\mathcal{B}$ *and satisfies* $\nabla \cdot \vec{\boldsymbol{J}} = \vec{\boldsymbol{0}}$.

(1) *For some smooth function* $f : \mathbb{R}^3 \to \mathbb{R}$, *prove that*

$$\int\int\int_{\mathcal{B}} (\nabla f) \cdot \vec{\boldsymbol{J}} \, dxdydz = 0.$$

(2) *Prove that*

$$\int\int\int_{\mathcal{B}} J_1 \, dxdydz = 0.$$

**Exercise 5.** *Let $\mathcal{D}$ denote the open unit disc in $\mathbb{R}^2$. Let $u$ be an eigenfunction for the Laplacian in $\mathcal{D}$; that is, a real-valued function of class $C^2$ defined in $\overline{\mathcal{D}}$, zero on the boundary, $\partial\mathcal{D}$, but not identically zero everywhere, such that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \lambda u,$$

*where $\lambda$ is a constant. Prove that*

$$\int\int_{\mathcal{D}} \left|\nabla u\right|^2 \, dxdy + \lambda \int\int_{\mathcal{D}} u^2 \, dxdy = 0.$$

**Exercise 6.** *The temperature at a point in a ball with conductivity $K$ is inversely proportional to the distance from the center of the ball. Find the rate of heat flow across a sphere $S$ of radius $a$ with center at the center of the ball.*