Math 53 - Multivariable Calculus

Take Home Assignment # 4

July 20th, 2011

Exercise 1. *Let* $\vec{\boldsymbol{F}}(x,y,z) = \left\langle e^y, xe^y, (z+1)e^z \right\rangle$. *Find a function* $f : \mathbb{R}^3 \to \mathbb{R}$ *such that* $\vec{\boldsymbol{F}}(x,y,z) = \nabla f$ *and then evaluate*

$$\int_c \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}},$$

where c *is the contour given by* $c : \vec{\boldsymbol{r}}(t) = \left\langle t, t^2, t^3 \right\rangle$, $t \in [0,1]$.

Exercise 2. *Evaluate*

$$\int_c (1 - ye^{-x})dx + e^{-x}dy,$$

where c is any path from $(0, 1)$ *to* $(1, 2)$

Exercise 3. *Argue whether or not the set* $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 \leq 1 \text{ or } 4 \leq x^2 + y^2 \leq 9\}$ *is open, connected and simply-connected. What about the torus* $T = S^1 \times S^1$*?*

Exercise 4. *Let D be the region bounded by a simple closed curve γ in the xy-plane. Prove that the coordinates of the centroid $(\bar{x}, \bar{y})$ of D are*

$$\bar{x} = \frac{1}{2A} \oint_\gamma x^2 dy, \qquad \bar{y} = -\frac{1}{2A} \oint_\gamma y^2 dx,$$

where A is the area of D.

Exercise 5. *Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a function with continuous second order partial derivatives such that: (i) $\phi_{xx} + \phi_{yy} + \phi_x = 0$ in the punctured plane $\mathbb{R}^2 \setminus \{(0,0)\}$, and (ii) $r\phi_x \to \frac{x}{2\pi r}$ and $r\phi_y \to \frac{y}{2\pi r}$ as $r = \sqrt{x^2 + y^2} \to 0$. Let C_R be the circle $x^2 + y^2 = R^2$. Show that the line integral*

$$\oint_{C_R} e^x(-\phi_y dx + \phi_x dy)$$

is independent of R, and evaluate it.

Exercise 6. *Suppose you have a vector field* $\vec{\boldsymbol{F}} = \langle M(x,y), N(x,y) \rangle$ *which has continuous first partial derivatives on* $\mathbb{R}^2 \setminus \{(a,b)\}$ *such that* $M_y = N_x$ *on all of* $\mathbb{R}^2 \setminus \{(a,b)\}$*. Prove that* $\vec{\boldsymbol{F}}$ *is conservative, or give a counterexample.*

Exercise 7.

(*i*) *Show that if* $u, v : \mathbb{R}^2 \to \mathbb{R}$ *are continuously differentiable on* $\mathbb{R}^2$ *and* $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$, *then* $u = \frac{\partial f}{\partial x}$ *and* $v = \frac{\partial f}{\partial y}$ *for some* $f : \mathbb{R}^2 \to \mathbb{R}$.

(*ii*) *Prove that there is no* $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ *such that*

$$\frac{\partial f}{\partial x} = \frac{-y}{x^2 + y^2} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{x}{x^2 + y^2}.$$