Exercise 1. *Let $P = (1,1,1)$, $Q = (0,3,1)$ and $R = (0,1,4)$.*

(a) (10 points) *Find the area of the triangle PQR.*

(b) (5 points) *Find the plane through P, Q, and R, expressed in the form $ax + by + cz = d$.*

(c) (5 points) *Is the line through $(1,2,3)$ and $(2,2,0)$ parallel to the plane in part (b)? Explain why or why not.*

(a) $\vec{PQ} = \langle -1, 2, 0 \rangle$, $\vec{PR} = \langle -1, 0, 3 \rangle$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = \langle 6, 3, 2 \rangle$$

So, area$(\Delta) = \frac{1}{2} |\vec{PQ} \times \vec{PR}| = \frac{1}{2}\sqrt{49} = 7/2$.

(b) A normal to the plane is given by $\vec{N} = \vec{PQ} \times \vec{PR} = \langle 6, 3, 2 \rangle$. Hence the equation has the form $6x + 3y + 2z = d$. To find d, we plug in the point $P = (1,1,1)$ since it's on the plane. This gives $d = 6 \cdot 1 + 3 \cdot 1 + 2 \cdot 1 = 11$. So, the eq$\underline{n}$ of the plane is $6x + 3y + 2z = 11$.

(c) The line is parallel to $\langle 2-1, 2-2, 0-3 \rangle = \langle 1, 0, -3 \rangle$.

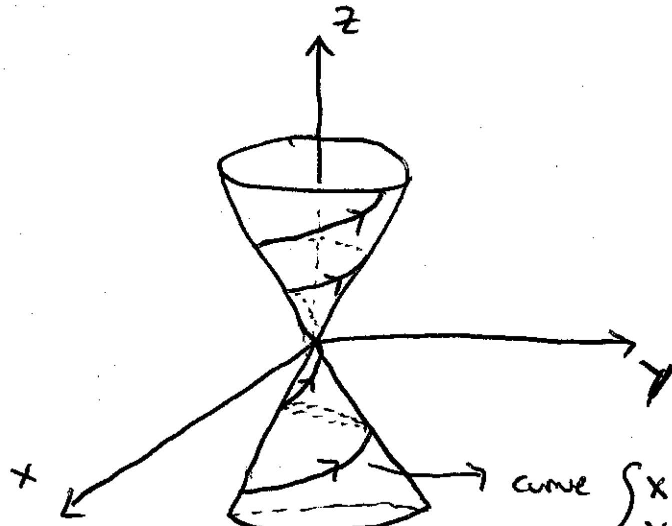
Since $\vec{N} \cdot \langle 1, 0, -3 \rangle = \langle 6, 3, 2 \rangle \cdot \langle 1, 0, -3 \rangle = 6 - 6 = 0$,

the line is parallel to the plane.

Exercise ~~1~~ 2. (10 *points*) *Show that the curve with parameteric equations* $x = t\cos(t)$, $y = t\sin(t)$, $z = t$ *lies on the cone* $z^2 = x^2 + y^2$, *and use this to help sketch the curve.*

If $x = t\cos(t)$, $y = t\sin(t)$, $z = t \Rightarrow x^2+y^2 = t^2(\cos^2(t)+\sin^2(t))$
$= t^2 = z^2$

$\Rightarrow x^2+y^2 = z^2$. So, the curve lies on the cone $z^2 = x^2+y^2$. Now, since $z = t$, the curve is a spiral on this cone.



$$\begin{cases} x(t) = t\cos(t) \\ y(t) = t\sin(t) \\ z(t) = t \end{cases}$$

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Exercise (10 *points*) *Find the linear approximation of the function* $f(x,y)=\sqrt{20-x^2-7y^2}$ *at* $(2,1)$ *and use it to approximate* $f(1.95, 1.08)$.

Let $f(x,y)=\sqrt{20-x^2-7y^2} \Rightarrow f_x = \dfrac{-x}{\sqrt{20-x^2-7y^2}}$ and

$f_y = \dfrac{-7y}{\sqrt{20-x^2-7y^2}}$. Hence $f_x(2,1) = -2/3$ and $f_y(2,1) = \frac{-7}{3}$.

Then the linear approximation of f at $(2,1)$ is given by

$f(x,y) \approx f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1)$

$\Rightarrow f(x,y) \approx -\frac{2}{3}x - \frac{7}{3}y + \frac{20}{3}$.

Hence, $f(1.95, 1.08) \approx -\frac{2}{3}(1.95) - \frac{7}{3}(1.08) + \frac{20}{3} = 2.846$

Exercise 4.

(a) (5 points) *Evaluate* $\int_c \vec{F} \cdot d\vec{r}$ *where* $\vec{F} = \langle e^y, xe^y + e^z, ye^z \rangle$ *and* c *is the line segment from* $(0, 2, 0)$ *to* $(4, 0, 3)$.

(b) (5 points) *Show that there is no vector field* $\vec{F}$ *such that* $\nabla \times \vec{F} = \langle 2x, 3yz, xz^2 \rangle$.

(c) (5 points) *If* c *is any piecewise-smooth simple closed plane curve and* f *and* g *are differentiable functions, show that* $\int_c f dx + g dy = 0$, $f = f(x)$, $g = g(y)$.

(a) $\vec{F} \doteq \langle e^y, xe^y + e^z, ye^z \rangle$ is conservative with potential $f^{\underline{n}}$

$f = xe^y + ye^z$. Indeed, $\nabla f = \langle e^y, xe^y + e^z, ye^z \rangle$.

So, by the F.T. line integrals, $\int_c \vec{F} \cdot d\vec{r} = f(4,0,3) - f(0,2,0)$

$= 4 - 2 = 2$.

(b) Suppose such an $\vec{F}$ existed. Then $\operatorname{div}(\operatorname{curl}(\vec{F})) = 2 + 3z - 2xz \neq 0$ which is a contradiction since $\operatorname{div}(\operatorname{curl}(\vec{F})) = 0$ $\forall$ vector fields on $\mathbb{R}^3$.

(c) For any piecewise-smooth simple close curve c in the plane bounding a region D, we can apply Green's theorem. So,

$$\int_c f\,dx + g\,dy = \iint_D \underbrace{\left[\frac{\partial}{\partial x} g(y) - \frac{\partial}{\partial y} f(x)\right]}_{=0} dA = 0.$$

$= 0$, since $\frac{\partial}{\partial x} g(y) = 0$ and $\frac{\partial}{\partial y} f(x) = 0$.

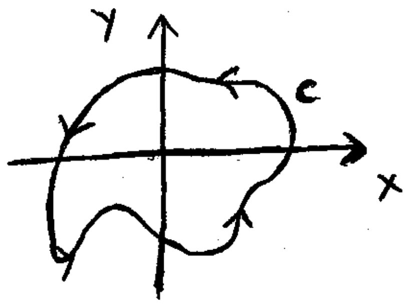
Exercise 5.

(a) (5 points) *Evaluate the surface integral* $\int\int_S \vec{F}\cdot\hat{n}dS$, *where* $\vec{F} = \langle xz, -2y, 3x\rangle$ *and* S *is the sphere* $x^2+y^2+z^2 = 4$ *with outward orientation.*

(b) (10 points) *Let*

$$\vec{F} = \frac{(2x^3+2xy^2-2y)\hat{\imath} + (2y^3+2x^2y+2x)\hat{\jmath}}{x^2+y^2}.$$

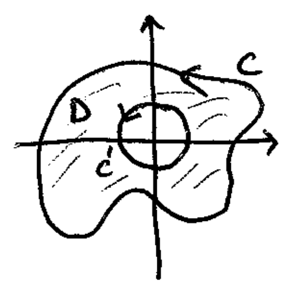
Evaluate $\int_c \vec{F}\cdot d\vec{r}$ *where c is shown below:*



(a) Since the sphere bounds a simple solid region, the divergence theorem applies and gives

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_E \operatorname{div}(\vec{F})\,dV = \iiint_E (z-2)\,dV = \iiint_E z\,dV - 2\iiint_E dV$$

$$= m\bar{z} - 2\left(\tfrac{4}{3}\pi 2^3\right) = -\frac{64}{3}\pi.$$



(b) Let c' be circle w/ center at origin and radius a. Let D be region bounded by c and c'. Then D's positively oriented boundary is $c \cup (-c')$. Hence, by Green,

$$0 = \iint_D (Q_x - P_y)\,dA = \int_c \vec{F}\cdot d\vec{r} + \int_{-c'} \vec{F}\cdot d\vec{r} \Rightarrow \int_c \vec{F}\cdot d\vec{r} = -\int_{-c'} \vec{F}\cdot d\vec{r} = \int_{c'} \vec{F}\cdot d\vec{r}.$$

$$\text{Now, } \int_{c'} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_0^{2\pi}\left[\frac{2a^3\cos^3(t) + 2a^3\cos(t)\sin^2(t) - 2a\sin(t)}{a^3}(-a\sin(t))\right.$$

$$\left. + \frac{2a^3\sin^3(t) + 2a^3\cos^2(t)\sin(t) + 2a\cos(t)}{a^2}(a\cos(t))\right]dt = \int_0^{2\pi}\frac{2a^2}{a^2}\,dt = 4\pi.$$

Exercise 6. (15 *points*) *The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on this ellipse that are nearest to and farthest from the origin.*

We need to find the extreme values of $f = x^2+y^2+z^2$ subject to the two constraints $g = x+y+2z = 2$ and $h = x^2+y^2-z=0$.

$\nabla f = \langle 2x, 2y, 2z \rangle$, $\lambda \nabla g = \langle \lambda, \lambda, 2\lambda \rangle$, $\mu \nabla h = \langle 2\mu x, 2\mu y, -\mu \rangle$.

Thus we need (1) $2x = \lambda + 2\mu x$, (2) $2y = \lambda + 2\mu y$, (3) $2z = 2\lambda - \mu$, (4) $x+y+2z=2$, and (5) $x^2+y^2-z=0$.

Now, (1), (2) $\Rightarrow$ $2(x-y) = 2\mu(x-y)$. So, if $x \neq y$, $\mu = 1$. Subing this into (3) gives $2z = 2\lambda - 1 \Rightarrow \lambda = z + 1/2$. Now, plugging $\mu = 1$ into (1) gives $2x = \lambda + 2x \Leftrightarrow \lambda = 0 \Rightarrow 0 = z + 1/2$ $\Rightarrow z = -1/2$. So, now (4) + (5) become $x+y-3=0$ and $x^2+y^2+\frac{1}{2}=0$ (note, $x^2+y^2+1/2=0$ only has $\mathbb{C}$ solns $\Rightarrow$ it gives no solns). Hence, $x=y$. Then (4) and (5) become $2x+2z=2$ and $2x^2-z=0 \Rightarrow z = 1-x$ and $z = 2x^2$. Thus,

$2x^2 = 1-x$ or $2x^2+x-1 = (2x-1)(x+1) = 0 \Rightarrow x = 1/2$ or $x = -1$.

Thus, the two points to check are $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $(-1,-1,2)$:

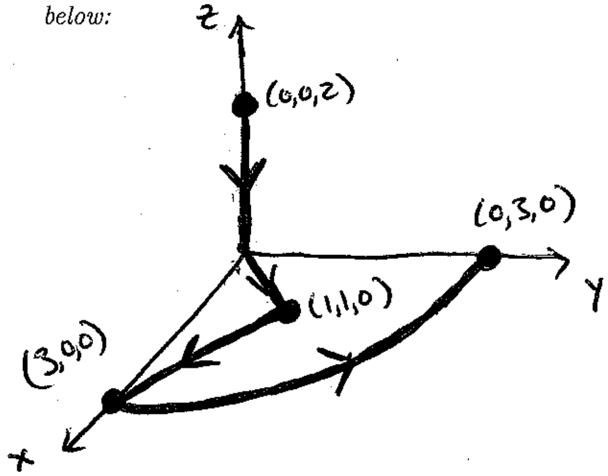
$f(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = \frac{3}{4}$ and $f(-1,-1,2) = 6 \Rightarrow (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is the point on the ellipse nearest $(0,0,0)$ and $(-1,-1,2)$ is the one farthest from $(0,0,0)$.

Exercise 7.

(a) (5 points) *If f is a harmonic function, that is $(\nabla\cdot\nabla)f = 0$, show that the line integral $\int_c f_y dx - f_x dy$ is path independent in any simply connected region D.*

(b) (5 points) *Use Stokes' theorem to evaluate $\int_c \vec{F}\cdot d\vec{r}$, where $\vec{F} = \langle xy, yz, zx\rangle$ and c is the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, oriented counter-clockwise as viewed from above.*

(c) (5 points) *Let $\vec{F} = \langle 3x^2yz - 3y, x^3z - 3x, x^3y + 2z\rangle$. Evaluate $\int_c \vec{F}\cdot d\vec{r}$, where c is the curve drawn below:*



(a) Let c be closed path in D, then by Green, $\oint_c f_y dx - f_x dy = -\iint_D (f_{xx} + f_{yy})\, dA = -\iint_D 0\, dA = 0.$

(b) The surface is given by $z = 1 - x - y$, $0 \le x \le 1$, $0 \le y \le 1 - x$.
So, $\hat{n}\,dS = \langle -f_x, -f_y, 1\rangle\, dx\,dy = \langle 1,1,1\rangle\, dx\,dy$. Also, $\mathrm{curl}(\vec{F}) = \langle -y, -z, -x\rangle$.
Hence, $\int_c \vec{F}\cdot d\vec{r} = \iint_S \mathrm{curl}(\vec{F})\cdot \hat{n}\, dS = \iint_D \langle -y, -z, -x\rangle \langle 1,1,1\rangle\, dx\,dy = -\iint_D dx\,dy = -\frac{1}{2}.$

(c) Because $\mathrm{curl}(\vec{F}) = 0$, $\vec{F}$ is conservative. We can find it's potential f^n: $f = x^3yz - 3xy + z^2$. Thus, by the F.T. for line integrals,

$$\int_c \vec{F}\cdot d\vec{r} = \int_c \nabla f\cdot d\vec{r} = f(0,3,0) - f(0,0,2) = 0 - 4 = -4.$$