Math 53 - Multivariable Calculus

Quiz # 9 Solns

April 13th, 2012

**Exercise 1.** *Compute $\int_c \vec{F}\cdot d\vec{r}$, where $\vec{F} = \langle x, y\rangle$ and c is given by $r(\vec{t}) = \left\langle \sqrt{\sqrt{\cos(t)}} - e^{\sqrt{\cos(t)}}, \sqrt{\sqrt{\sin(t)}} \right\rangle$ with $0 \le t \le \frac{\pi}{2}$. (Hint: Think about a "fundamental" theorem.)*

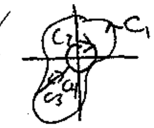
Note: $\vec{F} = \langle x,y\rangle = \nabla f$ where $f = \frac{x^2}{2} + \frac{y^2}{2}$. So, by the fundamental thm for line integrals $\int_c \vec{F}\cdot d\vec{r} = f\Big|_{\text{end pts.}}$. Now when $t=0$ $\sqrt{\sqrt{\cos(t)}} - e^{\sqrt{\cos(t)}} = 1-e$ and $\sqrt{\sqrt{\sin(t)}} = 0$ and when $t = \pi/2$ $\sqrt{\sqrt{\cos(t)}} - e^{\sqrt{\cos(t)}} = -1$ and $\sqrt{\sqrt{\sin(t)}} = 1$ $\Rightarrow$ end pts. are $(1-e, 0)$ and $(-1,1)$ $\Rightarrow \int_c \vec{F}\cdot d\vec{r} = \left(\frac{(-1)^2}{2} + \frac{(1)^2}{2}\right) - \left(\frac{(1-e)^2}{2} + \frac{0^2}{2}\right)$.

**Exercise 2.** *Compute $\int_c \vec{F}\cdot d\vec{r}$, where $\vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right\rangle$ and c is a circle of radius $r = 1$ centered at the origin $(0,0)$.*

We can parameterize c by $x(t) = \cos(t)$, $y(t) = \sin(t)$ $0 \le t \le 2\pi$. This gives

$$\oint_c \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \frac{\sin^2(t) + \cos^2(t)}{\sin^2(t) + \cos^2(t)}\,dt = \int_0^{2\pi} dt = 2\pi$$

**Exercise 3.** *Use Green's theorem to compute $\int_c \vec{F}\cdot d\vec{r}$, where $\vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right\rangle$ and c is ANY simple closed loop (i.e., a curve that starts and ends at the same point and it does not intersect itself) whose interior contains the origin $(0,0)$.*

Consider the curves  then $\int_{C_1 - C_2 + C_3 - C_4} = \int_{C_1 - C_2}$. Further the region enclosed by $C = C_1 - C_2 + C_3 - C_4$ is simply connected, so by Green's thm $\int_{C_1 - C_2} \vec{F}\cdot d\vec{r} = \oint \vec{F}\cdot d\vec{r}$

$= \iint_D \left[\frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2}\right] dA = 0 \Rightarrow \int_{C_1 - C_2} \vec{F}\cdot d\vec{r} = 0 \Rightarrow \int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$. Now,

$C = \partial D$

taking $C_2$ to be a circle ~~...~~ centered at $(0,0)$, we have (by question 2)

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r} = 2\pi.$$