Math 53 - Multivariable Calculus

Quiz # 1

January 25th, 2012

Solns

Exercise 1. *Find a polar equation for the curve represented in Cartesian coordinates by $x^2 + y^2 = 2cx$, where here $c \in \mathbb{R}$.*

$$x^2+y^2=2cx \iff r^2 = 2cr\cos(\theta) \iff r^2 - 2cr\cos(\theta) = 0 \iff r(r-2c\cos(\theta))=0$$

$\Rightarrow r=0$ or $r=2c\cos(\theta)$. However, when $\theta = \frac{\pi}{2}$ (or any integral multiple of $\pi/2$) $2c\cos(\theta)=0$. Hence, the curve is represented by the single equation $r = 2c\cos(\theta)$.

Exercise 2. *Show that the polar equation $r = a\sin(\theta) + b\cos(\theta)$, where $ab \neq 0$, represents a circle, and find its center and radius.*

$$r = a\sin(\theta) + b\cos(\theta) \Rightarrow r^2 = ar\sin(\theta) + br\cos(\theta) \Rightarrow x^2+y^2 = ay + bx \Rightarrow$$

$$x^2 - bx + y^2 - ay = 0 \Rightarrow x^2 - bx + y^2 - ay + \left(\tfrac{1}{2}b\right)^2 - \left(\tfrac{1}{2}b\right)^2 + \left(\tfrac{1}{2}a\right)^2 - \left(\tfrac{1}{2}a\right)^2 = 0$$

$$\Rightarrow x^2 - bx + \left(\tfrac{1}{2}b\right)^2 + y^2 - ay + \left(\tfrac{1}{2}a\right)^2 = \left(\tfrac{1}{2}b\right)^2 + \left(\tfrac{1}{2}a\right)^2 \Rightarrow \left(x - \tfrac{1}{2}b\right)^2 + \left(y - \tfrac{1}{2}a\right)^2 = \tfrac{1}{4}(a^2+b^2).$$

This is the equation of a circle with center $\left(\frac{1}{2}b, \frac{1}{2}a\right)$ and radius $\frac{1}{2}\sqrt{a^2+b^2}$.

Exercise 3. *find the area of the region that lies inside the curves $r = \sin(2\theta)$ and $r = \cos(2\theta)$. (Hint: $2\sin^2(2\theta) = 1 - \cos(4\theta)$)*

$$\sin(2\theta) = \cos(2\theta) \Rightarrow \frac{\sin(2\theta)}{\cos(2\theta)} = 1 \Rightarrow \tan(2\theta) = 1 \Rightarrow 2\theta = \frac{\pi}{4} \Rightarrow \theta = \pi/8$$

$$\Rightarrow A = \int_a^b \frac{1}{2}(f(\theta))^2\, d\theta = 2\cdot 8\int_0^{\pi/8} \frac{1}{2}\sin^2(2\theta)\, d\theta = 8\int_0^{\pi/8} \frac{1}{2}(1-\cos(4\theta))\, d\theta = \frac{\pi}{2} - 1$$

by symmetry