Math 54 - Linear Algebra and Differential Equations

Quiz # 5

April 27th, 2011

In this problem we will determine the mode expansion of the classical field representing a closed string moving through a D-dimensional spacetime. In non-supersymmetric bosonic string theory, the vector (field) which governs the propagation of strings is denoted by $\boldsymbol{X}(\tau,\sigma) = [X^1(\tau,\sigma), ..., X^D(\tau,\sigma)]$*, where* τ *and* σ *are the coordinates of the strings worldsheet. With a little bit of work (using certain symmetries present in string theory), one can show that each component of the vector obeys the following "wave" equation*

$$\left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right) X^\mu(\tau,\sigma) = 0; \tag{1}$$

here we use μ *to denote a component of* $\boldsymbol{X}(\tau,\sigma)$ *(hence,* $\mu = 1, ..., D$*). The wave equation is a seperable partial linear differential equation in terms of the variables* τ *and* σ*, so it has a solution of the form (for each specific* μ*)*

$$X^\mu(\tau,\sigma) = f(\sigma)g(\tau). \tag{2}$$

Exercise 1 (4 points)**.** *Apply this ansatz into the wave equation and show that the two functions must satisfy*

$$\frac{\partial^2 f(\sigma)}{\partial\sigma^2} = cf(\sigma), \qquad \frac{\partial^2 g(\tau)}{\partial\tau^2} = cg(\tau), \tag{3}$$

where c is an arbitrary constant.

Solution:

If we plug the ansatz $X^\mu(\tau,\sigma) = g(\tau)f(\sigma)$ into the wave equation (1), we obtain

$$\frac{f''(\sigma)}{f(\sigma)} = \frac{g''(\tau)}{g(\tau)}.$$

For this equation to hold for arbitrary values of τ, σ, it must be that both sides of this equation are equal to a constant, which is independent of τ, σ

$$\frac{f''(\sigma)}{f(\sigma)} = c = \frac{g''(\tau)}{g(\tau)}.$$

Namely,

$$f''(\sigma) = cf(\sigma),$$
$$g''(\tau) = c\,g(\tau).$$

Exercise 2 (2 points)**.** *Since we want to describe the mode expansion for closed strings we must make the assumption that* σ *is compact; that is, the functions* $X^\mu(\tau,\sigma)$ *must obey*

$$X^\mu(\tau,\sigma+\pi) = X^\mu(\tau,\sigma). \tag{4}$$

Write the most general solution to the first equation in equation (3) *and impose the boundary condition given by equation* (4) *to show that the constant* c *must take the values*

$$c = -4m^2, \tag{5}$$

where $m \in \mathbb{Z}$.

Solution:

If $c \neq 0$, the linearly independent solutions to $f''(\sigma) = cf(\sigma)$ are

$$e^{\pm\sqrt{c}\,\sigma}.$$

For this to be periodic under $\sigma \to \sigma + \pi$, we need

$$e^{\pm\sqrt{c}(\sigma+\pi)} = e^{\pm\sqrt{c}\,\sigma}.$$

For this to be true for any σ, we need

$$e^{\pm\pi\sqrt{c}} = 1,$$

namely $\pi\sqrt{c} = 2\pi im$, $m \in \mathbb{Z}$, $m \neq 0$ (note that this takes care of both signs in the previous equation). In other words,

$$c = -4m^2, \qquad m \in \mathbb{Z}, \quad m \neq 0.$$

On the other hand, if $c = 0$, the linearly independent solution to $f''(\sigma) = cf(\sigma)$ are

$$1, \quad \sigma.$$

The second one is not periodic under $\sigma \to \sigma + \pi$. So, only the first one is appropriate.

Combining the $c \neq 0$ and $c = 0$ cases, we have

$$c = -4m^2, \quad m \in \mathbb{Z}, \qquad f(\sigma) = e^{\pm 2im\sigma}$$

Exercise 3 (4 points)**.** *With the knowledge from question* (2) *one can show that the linearly indepedent solutions for* $f(\sigma)$ *and* $g(\tau)$ *are as follows: (i)* $f(\sigma) = e^{\pm 2im\sigma}$*, where* $m \in \mathbb{Z}$*, and (ii)* $g(\tau) = e^{\pm 2im\tau}$ *when* $m \in \mathbb{Z}$*,* $m \neq 0$*, while* $g(\tau) = 1, \tau$ *when* $m = 0$*. Use this to show that the most general solution to equation* (1) *is of the form*

$$X^\mu(\tau,\sigma) = x^\mu + a^\mu\tau + \sum_{m\neq 0}\left(b^\mu_m e^{-2im(\tau-\sigma)} + \tilde{b}^\mu_m e^{-2im(\tau+\sigma)}\right), \tag{6}$$

where $x^\mu, a^\mu, b^\mu_m, \tilde{b}^\mu_m$ *are arbitrary constants.*

Solution:

By solving $g''(\tau) = cg(\tau)$ for the values of m given in b), we obtain

$$\begin{aligned} &c = -4m^2, \qquad m \in \mathbb{Z}, \quad m \neq 0 \qquad &\Longrightarrow \qquad &g(\tau) = e^{\pm 2im\tau}, \\ &c = 0 &\Longrightarrow \qquad &g(\tau) = 1,\ \tau. \end{aligned}$$

Multiplying f and g, we conclude that $X^\mu(\tau,\sigma)$ is given by a linear combination of the following functions:

$$e^{-2im(\tau+\sigma)}, \quad e^{-2im(\tau-\sigma)}, \quad 1, \quad \tau,$$

with $m \in \mathbb{Z}$, $m \neq 0$. Therefore, the most general solution to the wave equation (1) satisfying the periodicity (4) is

$$X^{\mu}(\tau,\sigma) = x^{\mu} + a^{\mu}\tau + \sum_{m\neq 0}\left(b^{\mu}_{m}e^{-2im(\tau-\sigma)} + \tilde{b}^{\mu}_{m}e^{-2im(\tau+\sigma)}\right),$$

where the coefficients $x^{\mu}, a^{\mu}, b^{\mu}_{m}, \tilde{b}^{\mu}_{m}$ are constants.

The expression in equation (6) *is called the mode expansion of the field* $X^{\mu}(\tau,\sigma)$. *Upon quantization of the theory, these modes are realized as particles. For instance, one of the vibration modes of the quantized closed string corresponds to the graviton - the particle which carries the gravitational force.*