Math 54 - Linear Algebra and Differential Equations

Quiz # 3

March 16th, 2011

Exercise 1 (2 points)**.**

Let V be a vector space over $\mathbb{R}$, an inner product on V is a map $\langle \cdot, \cdot \rangle \longrightarrow \mathbb{R}$ such that, for all $x, y, z \in V$ and $\alpha, \beta \in \mathbb{R}$,

(*i*) $\langle x, y \rangle = \langle y, x \rangle$,

(*ii*) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$,

(*iii*) $\langle x, x \rangle \geq 0$ with equality only for $x = 0$.

One can check that the 'dot' product, $\langle x, z \rangle = x \cdot z = \sum_{i=1}^{n} \alpha_i \beta_i$, introduced in class is AN inner product, but not the only one. In particular, show that $\langle \cdot, \cdot \rangle \longrightarrow \mathbb{R}$, defined by

$$\langle f, g \rangle = \int_0^1 f(x) g(x) dx,$$

is an inner product on $C(\mathbb{R})$, the vector space of all continuous real-valued functions.

Solution: We have:

(i) $\langle f, g \rangle = \int_0^1 f(x)g(x)dx = \int_0^1 g(x)f(x)dx = \langle g, f \rangle$, since multiplication between real-valued functions is commutative.

(ii) $\langle \alpha f + \beta g, h \rangle = \int_0^1 (\alpha f(x) + \beta g(x))h(x)dx = \int_0^1 \alpha f(x)h(x)dx + \int_0^1 \beta g(x)h(x)dx = \alpha \int_0^1 f(x)h(x)dx + \beta \int_0^1 g(x)h(x)dx = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$.

(iii) $\langle f, f \rangle = \int_0^1 f(x)f(x)dx = \int_0^1 f(x)^2 dx > 0$ unless $f(x) = 0$ for $0 \leq x \geq 1$.

Exercise 2 (1 point)**.** *As we have seen, the collection* $\{1, x, x^2\}$ *gives a basis for* $\mathcal{P}_2(\mathbb{R})$*. However, this basis is not orthonormal with respect to the integral inner product defined in exercise 1. Another basis for* $\mathcal{P}_2(\mathbb{R})$ *is given by* $\{1, \sqrt{3}(-1 + 2x), \sqrt{5}(1 - 6x + 6x^2)\}$*. Show that this basis is orthonormal with respect to the inner product* $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$*, where* $f, g \in \mathcal{P}_2(\mathbb{R})$*.*

Solution: Denote $e_1 = 1$, $e_2 = \sqrt{3}(-1+2x)$, $e_3 = \sqrt{5}(1-6x+6x^2)$. From elementary calculus we see

$$\langle e_1, e_1 \rangle = \int_0^1 1^2 dx = 1,$$
$$\langle e_1, e_2 \rangle = \int_0^1 1 \cdot \sqrt{3}(-1+2x) dx = 0,$$
$$\langle e_1, e_3 \rangle = \int_0^1 1 \cdot \sqrt{5}(1-6x+6x^2) dx = 0,$$

and likewise for e_2 and e_3.

Exercise 3 (2 points). *Let $W \subset \mathbb{R}^3$ be the subspace $W = span_{\mathbb{R}}\{e_1, e_2\}$ and let $H \subset \mathbb{R}^3$ be the subspace $H = span_{\mathbb{R}}\{e_3\}$. Show that, with respect to the dot product $v \cdot w = \sum_{i=1}^3 \alpha_i \beta_i$, W and H are orthogonal complememnts. Also, show that any vector $v \in \mathbb{R}^3$ can be written as $v = v_W + v_H$, where $v_W \in W$ and $v_H \in H$.*

Solution: To show that W and H are orthogonal, consider $x \cdot y$ with $x \in W$ and $y \in H$. Since $W = \text{span}_{\mathbb{R}}\{e_1, e_2\}$ and $H = \text{span}_{\mathbb{R}}\{e_3\}$ we can write $x = \alpha_1 e_1 + \alpha_2 e_2 + 0e_3$ and $y = 0e_1 + 0e_2 + \beta_3 e_3$. Thus, $x \cdot y = \sum \alpha_i \beta_i = \alpha_1 \cdot 0 + \alpha_2 \cdot 0 + 0 \cdot \beta_3 = 0$. Finally, let $v \in \mathbb{R}^3$ then we can write $v = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3$. So, setting $v_W = \gamma_1 e_1 + \gamma_2 e_2$ and $v_H = \gamma_3 e_3$, we see that $v = v_W + v_H$ with $v_W \in W$ and $v_H \in H$.

Exercise 4 (1 point). *Fill in the following sentence: Let $U \subseteq V$, a projection P is a linear transformation $P : V \longrightarrow U \subseteq V$ such that...*

Solution: A a projection P is a linear transformation $P : V \longrightarrow U$ such that $P \circ P = P$.

Exercise 5 (2 points). *Find the eigenvalues of the projection operator $P : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}.$$

Solution: With respect to the usual basis $\mathcal{B} = \{e_1, e_2, e_3\}$, we have

$$[P]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

And so, we can immediately read off the eigenvalues; namely 1, 1, and 0.

Exercise 6 (2 points)**.** *Find* e^A*, where* $A = \begin{pmatrix} 1 & 2 \\ 0 & 5 \end{pmatrix}$.

Solution: Since A is (upper) triangular, its eigenvalues are given by the components on the diagonal. So, we have $\lambda_1 = 1$ and $\lambda_2 = 5$. Because A has $\dim(\mathbb{R}^2) = 2$ distinct eigevalues we know that A is diagonalizable (since the corresponding eigenvectors are linearly independent they give a basis). Further, the corresponding eigenvectors are $v_{\lambda_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_{\lambda_2} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. So, we can write $A = PDP^{-1}$, where $P = [v_{\lambda_1}\ v_{\lambda_2}]$ and $D = \text{diag}(1, 5)$. Now, we have

$$\begin{aligned}
e^A &= \sum_{n=0}^{\infty} \frac{A^n}{n!} \\
&= I + \frac{A}{1} + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots \\
&= I + \frac{(PDP^{-1})}{1} + \frac{(PDP^{-1})^2}{2!} + \frac{(PDP^{-1})^3}{3!} + \cdots \\
&= I + \frac{PDP^{-1}}{1} + \frac{PDP^{-1}PDP^{-1}}{2!} + \frac{PDP^{-1}PDP^{-1}PDP^{-1}}{3!} + \cdots \\
&= I + \frac{PDP^{-1}}{1} + \frac{PD^2P^{-1}}{2!} + \frac{PD^3P^{-1}}{3!} + \cdots \\
&= PP^{-1} + \frac{PDP^{-1}}{1} + \frac{PD^2P^{-1}}{2!} + \frac{PD^3P^{-1}}{3!} + \cdots \\
&= P\left(I + \frac{D}{1} + \frac{D^2}{2!} + \frac{D^3}{3!} + \cdots\right)P^{-1} \\
&= \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} 1^2 & 0 \\ 0 & 5^2 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} 1^3 & 0 \\ 0 & 5^3 \end{pmatrix} + \cdots\right)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^1 & 0 \\ 0 & e^5 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \\
&= \begin{pmatrix} e^5 & e^1 - e^5 \\ 0 & e^1 \end{pmatrix}.
\end{aligned}$$

Exercise 7 (BONUS 2pts)**.** *Prove that if* $T : V \longrightarrow V$ *has* $\dim(V)$ *distinct eigenvalues, then* T *has a* *diagonal matrix with respect to some basis of V.*

Solution: Suppose $T : V \longrightarrow V$ has $\dim(V)$ distinct eigenvalues, say $\lambda_1, ..., \lambda_{\dim(V)}$. Now, for each $j \in \{1, ..., \dim(V)\}$, let v_j be a nonzero eigenvector corresponding to λ_j. Because nonzero eigenvectors corresponding to distinct eigenvalues are linearly independent, the set of vectors $(v_1, ..., v_{\dim(V)})$ is linearly independent. Further, a linearly independent list of $\dim(V)$ vectors gives a basis of V. Hence, $(v_1, ..., v_{\dim(V)})$ is a basis of V. And so, with respect to this basis, T has a diagonal matrix.