Math 54 - Linear Algebra and Differential Equations

Quiz # 1

February 2nd, 2011

Exercise (2 points). *True or False: If the set of vectors* $\{v_1, v_2, v_3, v_4, v_5\}$ *spans* $\mathbb{R}^4$*, then one can always delete a vector from this set, leaving a set of four vectors which also spans* $\mathbb{R}^4$*.*

Solution:

This is true. Indeed, if the set $S = \{v_1, ..., v_5\}$ spans $\mathbb{R}^4$ then at least four of the vectors in S must be linearly independent. Without loss of generality, let's assume that the first four vectors v_1, v_2, v_3, v_4 are linearly independent with respect to each other. Now, the remaining fifth vector v_5 cannot be linearly independent from the other four, otherwise you would have five linearly independent vectors in a 4-dimensional space. Hence, this fifth vector v_5 must be linearly dependent to at least one of the vectors in the linearly independent set. Additionally, if v_5 is linearly dependent to some (or all) of the vectors $v_1, ..., v_4$, then

$$\text{span}_{\mathbb{R}}\big\{v_1, v_2, v_3, v_4, v_5\big\} = \text{span}_{\mathbb{R}}\big\{v_1, v_2, v_3, v_4\big\}.$$

Finally, note that since the four vectors $v_1, ..., v_4$ are linearly independent they span $\mathbb{R}^4$. Hence,

$$\text{span}_{\mathbb{R}}\big\{v_1, v_2, v_3, v_4\big\} = \mathbb{R}^4.$$

Exercise (2 points). *An affine transformation* $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *is defined by* $T(x) = Ax + b$*, with* $A \in M_{m\times n}(\mathbb{R})$ *and* $b \in \mathbb{R}^m$*. Show that* T *is not a linear transformation when* $b \neq 0$*.*

Solution:

Suppose $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation. Then, by definition of linearity, we must have that $L(0_{\mathbb{R}^n}) = 0_{\mathbb{R}^m}$. However, in our case, $T(0_{\mathbb{R}^n}) = A0_{\mathbb{R}^m} + b = b$. Consequently, we see that if $b \neq 0_{\mathbb{R}^m}$ then T cannot be linear.

Exercise. *Let* $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ *be the linear transformation defined by*

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \longmapsto \begin{pmatrix} x_2 + x_3 + x_4 \\ x_3 + x_4 \\ x_4 \\ 0 \end{pmatrix}$$

(a) (2 points) Find the matrix representation of T*,* $[T]_{\mathcal{B}}$*, with respect to the usual basis of* $\mathbb{R}^4$*,* $\mathcal{B} = \{e_i\}_{i=1}^4$*.*
(b) (2 points) Use this to show that T *is a nilpotent operator; that is, show that there exists an* $n \in \mathbb{N}$ *such that taking the* n^{th} *power of* $[T]_{\mathcal{B}}$ *results in the zero matrix,*

$$\underbrace{[T]_{\mathcal{B}}[T]_{\mathcal{B}} \cdots [T]_{\mathcal{B}}}_{n \text{ times}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(c) (2 points) Use the previous results to find $\cos\left([T]_{\mathcal{B}}\right)$*, where* $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$.

Solution: (a) In order to find the matrix representation of T, we need to see how T acts on the basis vectors $\{e_1, ..., e_4\}$. So, consider:

$$T(e_1) = T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$T(e_2) = T\left(\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$T(e_3) = T\left(\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$T(e_4) = T\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

Now, the matrix representation of T, with respect to the basis $\mathcal{B} = \{e_1\}_{i=1}^4$, is the matrix $[T]_{\mathcal{B}}$ whose j^{th}

column is given by $T(e_j)$. Thus,

$$[T]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(b) To find n we simply start multiplying:

$$[T]_{\mathcal{B}}[T]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$[T]_{\mathcal{B}}[T]_{\mathcal{B}}[T]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$[T]_{\mathcal{B}}[T]_{\mathcal{B}}[T]_{\mathcal{B}}[T]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, we can see that $([T]_{\mathcal{B}})^4 = 0$, and hence $T^4 = 0$.

Note, we could have seen this directly from the definition of T. Namely, T shifts the components of a vector up by one, and replaces the bottom component with a zero. Hence, after applying T four times to any vector, we will have replaced every component with a zero; that is, $T^4 = 0$.

(c) We have

$$\begin{aligned} \cos\big([T]_{\mathcal{B}}\big) &= \sum_{n=0}^{\infty} \frac{(-1)^n([T]_{\mathcal{B}})^{2n}}{(2n)!} \\ &= ([T]_{\mathcal{B}})^0 - \frac{1}{2}([T]_{\mathcal{B}})^2 + \frac{1}{4}([T]_{\mathcal{B}})^4 - \frac{1}{6}([T]_{\mathcal{B}})^6 + \cdots. \end{aligned}$$

Now, from part (b) we know that $([T]_{\mathcal{B}})^n = 0$ for all $n \geq 4$. Hence, the above infinite sum reduces to

$$\begin{aligned} \cos\big([T]_{\mathcal{B}}\big) &= ([T]_{\mathcal{B}})^0 - \frac{1}{2}([T]_{\mathcal{B}})^2, \\ &= 1_4 - \frac{1}{2}([T]_{\mathcal{B}})^2, \end{aligned}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$= \begin{pmatrix} 1 & 0 & -\frac{1}{2} & -1 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$