# Math 53 - Multivariable Calculus

Quiz # 12

Solns

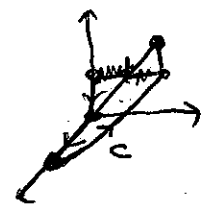
December 1st, 2011

**Exercise 1.** *Let $c$ be a simple, closed smooth curve and let $\Sigma_1$ and $\Sigma_2$ be two smooth surfaces (with positive orientation) with $\partial S_1 = \partial S_2 = c$. Additionally, let $\vec{F}$ be a vector field on $\mathbb{R}^3$ whose components have continuous partial derivatives. Explain why or why not $\int\int_{\Sigma_1} curl(\vec{F}) \cdot dS = \int\int_{\Sigma_2} curl(\vec{F}) \cdot dS$.*

We have that $\Sigma_1, \Sigma_2$ and $\vec{F}$ satisfy Stokes' theorem. Hence

$$\iint_{\Sigma_1} \text{curl}(\vec{F}) \cdot d\vec{S} = \oint_c \vec{F} \cdot d\vec{r} = \iint_{\Sigma_2} \text{curl}(\vec{F}) \cdot d\vec{S}$$

**Exercise 2.** *Suppose a particle moves along line segments from the origin to the points $(1, 0, 0)$, $(1, 2, 1)$, $(0, 2, 1)$, and back to the origin under the influence of the force field $\vec{F} = \langle z^2, 2xy, 4y^2 \rangle$. Find the work done. (HINT: Use Stokes' theorem)*

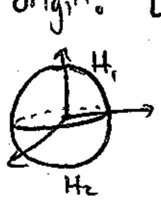



Let $S$ be the planar region enclosed by the path of the particle; i.e, $S$ is the portion of the plane $z = \frac{1}{2}y$, $0 \le x \le 1$, $0 \le y \le 2$ with upward orientation. Also, $\text{curl}(\vec{F}) = \langle 8y, 2z, 2y \rangle$. Thus,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \iint_S \text{curl}(\vec{F}) \cdot \hat{n}\, dS = \int_0^1 \int_0^2 \left(2y - \tfrac{1}{2}y\right) dy\, dx = 3.$$

**Exercise 3.** *Let $S_a^2$ denote the sphere of radius $a$ and suppose the vector field $\vec{F}$ satisfies the hypotheses of Stokes' theorem. Show that $\int\int_{S_a^2} curl(\vec{F}) \cdot d\vec{S} = 0$.(HINT: Decompose $S_a^2$ as the union of the northern hemisphere and the southern hemisphere. Then apply Stokes' theorem on each piece (cough careful about orientations cough)).*

The problem is clearly translational invariant. So, WLOG, assume $S_a^2$ centered at origin. Let $H_1$ and $H_2$ be the northern and southern hemisphere, respectively.




Then $$\iint_{S_a^2} \text{curl}(\vec{F}) \cdot d\vec{S} = \iint_{H_1} \text{curl}(\vec{F}) \cdot d\vec{S} + \iint_{H_2} \text{curl}(\vec{F}) \cdot d\vec{S} = \oint_{C_1} \vec{F} \cdot d\vec{r} + \oint_{C_2} \vec{F} \cdot d\vec{r},$$

where $C_i = \partial H_i$. But $C_1$ is $x^2 + y^2 = a^2$ oriented in the counterclockwise direction while $C_2$ is $x^2 + y^2 = a^2$ in the clockwise direction. Thus,

$$\iint_{S_a^2} \text{curl}(\vec{F}) \cdot d\vec{S} = \oint_{C_1} \vec{F} \cdot d\vec{r} + \oint_{C_2} \vec{F} \cdot d\vec{r} = \oint_{C_1} \vec{F} \cdot d\vec{r} - \oint_{C_1} \vec{F} \cdot d\vec{r} = 0.$$