Math 53 - Multivariable Calculus

Solns

Quiz # 10

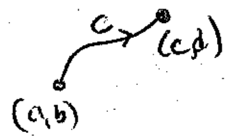
November 10th, 2011

Exercise 1. *Let $\vec{F} = \langle e^x \sin(y), e^x \cos(y)\rangle$. Find a potential function for $\vec{F}$ and then evaluate $\int_C \vec{F}\cdot d\vec{r}$ where C is ANY path from (a,b) to (c,d).*

$$\nabla f = \vec{F} \iff f_x = e^x\sin(y) \xrightarrow{\int dx} f = e^x\sin(y) + g(y) \xrightarrow[\text{w/ } f_y]{\frac{\partial f}{\partial y} \text{ + compare}} g'(y) = 0 \Rightarrow f(x,y) = e^x\sin(y) + K.$$

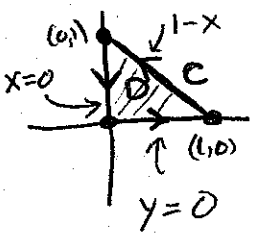
$$f_y = e^x\cos(y)$$

Since $\vec{F} = \nabla f$, by the fundamental thm for line integrals, if C is any path from (a,b) to (c,d)



$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(c,d) - f(a,b)$$
$$= e^c\sin(d) - e^a\sin(b)$$

Exercise 2. *Use Green's theorem to find the work done by the force $\vec{F} = \langle x(x+y), xy^2\rangle$ in moving a particle from the origin along the x-axis to $(1,0)$, then along the line segment to $(0,1)$, and then back to the origin along the y-axis.*



By Green, $W = \int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA \Rightarrow \int_C x(x+y)\,dx + xy^2\,dy =$

$$= \iint_D (y^2 - x)\,dy\,dx = \int_0^1\int_0^{1-x} (y^2 - x)\,dy\,dx = \int_0^1 \tfrac{1}{3}y^3 - xy\Big|_0^{1-x} dx$$

$$= \int_0^1 \left(\tfrac{1}{3}(1-x)^3 - x(1-x)\right)dx = -\tfrac{1}{12}(1-x)^4 - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3\Big|_0^1$$

$$= -\frac{1}{12}$$

Exercise 3. *Is there a vector field $\vec{F}$ on $\mathbb{R}^3$ such that $\nabla\times\vec{F} = \langle x\sin(y), \cos(y), z - xy\rangle$? Explain why or why not!*

No! Suppose such a $\vec{F}$ existed, then $\operatorname{div}(\nabla\times\vec{F}) = \frac{\partial}{\partial x}(x\sin(y)) + \frac{\partial}{\partial y}(\cos(y)) + \frac{\partial}{\partial z}(z - xy)$

$$= \sin(y) - \sin(y) + 1$$
$$= 1 \neq 0$$

This contradicts theorem 11; namely, if $\vec{F} = \langle P,Q,R\rangle$ and $P,Q,R \in C^2(\mathbb{R}^3)$ then $\operatorname{div}(\nabla\times\vec{F}) = 0$.