Math 53 - Multivariable Calculus

Practice Midterm # 2

November 17th, 2011

Solns

Exercise 1. *Evaluate* $\int\int_R e^{x+y}\, dA$*, where* R *is given by the inequality* $|x| + |y| \leq 1$.

Let $u = x+y$, $v = -x+y \Rightarrow u+v = 2y \Rightarrow y = \frac{1}{2}(u+v)$ and $x = \frac{1}{2}(u-v)$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix} = \frac{1}{2}.$$ Now, $|u| = |x+y| \leq |x|+|y| \leq 1 \Rightarrow -1 \leq u \leq 1$

$v = |-x+y| \leq |x| + |y| \leq 1 \Rightarrow -1 \leq v \leq 1$

$$\iint_R e^{x+y}\, dA = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} e^u\, du\, dv = e - \frac{1}{e}$$

Exercise 2. *Let* $\vec{F} = \nabla f$*, where* $f(x,y) = \sin(x-2y)$*. Find a curve* c *such that* $\int_c \vec{F} \cdot d\vec{r} = 1$.

$\vec{F} = \nabla f \Rightarrow \vec{F}$ conservative $\Rightarrow \int_c \vec{F} \cdot d\vec{r} = f\big|_{\text{end pts}}$

since $f(0,0) = \sin(0) = 0$ and $f(\frac{\pi}{2}, 0) = \sin(\pi/2) = 1$

one possible curve is the straight line from $(0,0)$ to $(\pi/2, 0)$, $\vec{r}(t) = \langle \frac{\pi}{2}t, 0 \rangle$ $0 \leq t \leq 1$.

Exercise 3. *Suppose you have a vector field $\vec{F} = \langle M(x,y), N(x,y)\rangle$ which has continuous first partial derivatives on $\mathbb{R}^2 \setminus \{(a,b)\}$ such that $M_y = N_x$ on all of $\mathbb{R}^2 \setminus \{(a,b)\}$. Prove that $\vec{F}$ is conservative, on $\mathbb{R}^2$ or give a counterexample.*

This is false. Consider, for example, the vector field $\vec{F} = \dfrac{\langle -y, x\rangle}{x^2+y^2}$.

This has C^1 components on $\mathbb{R}^2 \setminus \{(0,0)\}$ and $M_y = N_x$ on $\mathbb{R}^2 \setminus (0,0)$

However, on C , $\oint_C \vec{F}\cdot d\vec{r} = 2\pi \neq 0$, as we've seen $\Rightarrow$ $\vec{F}$ not conservative on $\mathbb{R}^2$

Exercise 4. *Let $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$ denote the unit sphere in $\mathbb{R}^3$. Evaluate the following surface integral over S:*

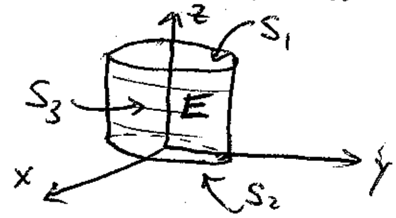
$$\iint_S (x^2+y+z)\, dA.$$

$\hat{n} = \langle x, y, z\rangle$. Now let $\vec{F} = \langle x, 1, 1\rangle \Rightarrow \vec{F}\cdot\hat{n} = x^2+y+z$, and $\operatorname{div}(\vec{F}) = 1$. Hence,

$$\iint_S (x^2+y+z)\, dA = \iint_S \vec{F}\cdot\hat{n}\, dA = \iiint_{x^2+y^2+z^2\le 1} \operatorname{div}(\vec{F})\, dV$$

$$= \iiint_{x^2+y^2+z^2\le 1} dV = \frac{4}{3}\pi$$

Exercise 5. *Let M be the solid cylinder $x^2+y^2 \le 1$, $0 \le z \le 1$ and let $\vec{F} = \langle xy, yz, zx\rangle$. Verify that the divergence theorem is true for $\vec{F}$ on the region M.*



$\text{div}(\vec{F}) = y + z + x$, so

$$\iiint_E \text{div}(\vec{F})\, dV = \int_0^{2\pi}\int_0^1\int_0^1 (r\cos(\theta) + r\sin(\theta) + z)\, r\, dz\, dr\, d\theta = \pi/2$$

$\boxed{\iint_{S_1}}$ here $z=1$, $\hat{n} = \hat{k}$, $\vec{F} = \langle xy, y, x\rangle$

$$\iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot\hat{n}\, dS = \iint_{S_1} x\, dS = \int_0^{2\pi}\int_0^1 r\cos(\theta)\, r\, dr\, d\theta = 0$$

$\boxed{\iint_{S_2}}$ $z=0$, $\hat{n} = -\hat{k}$, $\vec{F} = \langle xy, 0, 0\rangle$

$$\iint_{S_2} \vec{F}\cdot d\vec{S} = \iint_{S_2} 0\, dS = 0$$

$\boxed{\iint_{S_3}}$ parameterize S_3 by $\vec{r}(\theta, z) = \langle \cos(\theta), \sin(\theta), z\rangle$, $0 \le \theta \le 2\pi$

$0 \le z \le 1$. Here $\vec{r}_\theta \times \vec{r}_z = \langle \cos(\theta), \sin(\theta), 0\rangle$

$$\iint_{S_3} \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(\vec{r}_\theta \times \vec{r}_z)\, dA = \int_0^{2\pi}\int_0^1 (\cos^2(\theta)\sin(\theta) + z\sin^2(\theta))\, dz\, d\theta$$

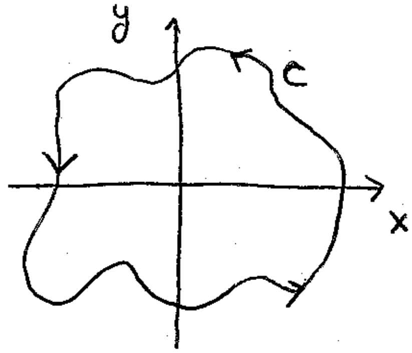
$$= \pi/2$$

$$\Rightarrow \iint_S = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} = 0 + 0 + \pi/2 = \pi/2$$

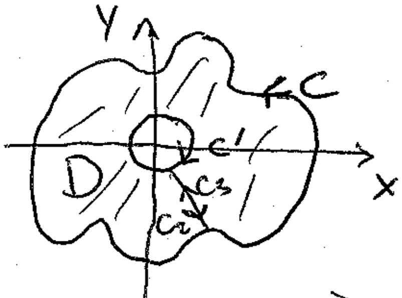
Exercise 6. (*Very Hard!*) *Let*

$$\vec{F} = \frac{(2x^3+2xy^2-2y)\hat{\imath} + (2y^3+2x^2y+2x)\hat{\jmath}}{x^2+y^2}.$$

Evaluate $\int_c \vec{F}\cdot d\vec{r}$ *where c is shown below:*



We don't know how to parametrize c, but we can use Green's theorem to relate $\oint_c \vec{F}\cdot d\vec{r}$ to $\oint_{c'} \vec{F}\cdot d\vec{r}$ where c' is unit circle centered at origin (as in class !!!).



$\gamma = C - C' + C_3 - C_2$ encloses a simple region D and on D $\vec{F}$ is conservative $\Rightarrow \oint_\gamma \vec{F}\cdot d\vec{r} = 0$. Further, $\int_{C_3} \vec{F}\cdot d\vec{r} - \int_{C_3} \vec{F}\cdot d\vec{r} = 0$. Hence, $\oint_c \vec{F}\cdot d\vec{r} = \oint_{c'} \vec{F}\cdot d\vec{r}$. So, it suffices to compute $\oint_{c'} \vec{F}\cdot d\vec{r}$. So,

$$\oint_c \vec{F}\cdot d\vec{r} = \oint_{c'} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt = \int_0^{2\pi}\left[\frac{2\cos^3(t)+2\cos(t)\sin^2(t)}{1^3} + -2\sin(t)\big(-\sin^2(t)\big) + \frac{2\sin^3(t)+2\cos^2(t)\sin(t)+2\cos(t)}{1^2}\big(\cos(t)\big)\right]dt$$

$$= \int_0^{2\pi} 2\,dt = 4\pi.$$