Classification of CS actions and WZW terms

Chern–Simons Actions

To begin: let $G \hookrightarrow E \longrightarrow M$ be principal G-bundle, G connected, simply connected compact, $\dim(M) = 3$.

Fix connection ω on E, and assume E trivial. In this case, can pull ω down to global $\mathfrak{g}$-valued 1-form on M,

$$A = s^*(\omega)$$

global section $s : M \longrightarrow E$

Hence, we can integrate A over M, and we can define the CS functional by the familiar formula

ad$_G$-invariant symm. bilinear form

$$S(A) = \frac{k}{8\pi^2} \int_M \mathrm{Tr}\left(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A\right)$$

The parameter k must take integer values so that the quantum measure $e^{2\pi i S(A)}$ is gauge invariant.

$\pi_0(G)=0$

<u>Observations</u>: If G is connected, simply connected and compact, then any G-

$\pi_1(G)=0$ $\qquad$ $\pi_2(G)=0$

bundle $G \hookrightarrow E \to M$ over a three mnfld is trivial, and so our definition of the CS action is adequate.

Trivializing E: construct $s: M \to E$ inductively over $M^0, M^1, M^2, M^3 = M$, obstruction to extending over M^i lies in $\pi_{i-1}(G)$.

For more general G, can have nontrivial E and in this case need to modify the def$^{\underline{n}}$ of $S(A)$ since ω no longer can be represented by a global $\mathfrak{g}$-valued 1-form on M

(can get local $A_\alpha = s_\alpha^*(\omega)$ on open cover U_α, but on intersection $U_\alpha \cap U_\beta$, A_α doesn't transform in the adjoint rep of $G \Rightarrow$ integrand $\mathrm{Tr}(A dA + \frac{2}{3}A^3)$ can't be glued together to give global 3-form on $M \Rightarrow$ can't integrate $\mathrm{Tr}(\cdots)$ in this case)

So, we need to generalize our previous def$^{\underline{n}}$ of $S(A)$.

<u>Warm-up</u>: Let $G \hookrightarrow E \to M$ be trivial. From cobordism theory, $\exists$ 4-mnfld B s.t. $\partial B = M$. Also, can extend E to E' where $E'|_{\partial B} = E$, and can extend ω to ω' on E'. Further, $d\,\mathrm{Tr}(A\,dA + \frac{2}{3}A^3) = \mathrm{Tr}(F \wedge F)$, F curvature of A, so can write

$$S(A) = \frac{k}{8\pi^2} \int_B \mathrm{Tr}(F' \wedge F')$$

F' curvature any gauge field A' on B which restricts to A on ∂B.

This reduces to original def$^{\underline{n}}$ of $S(A)$ when it makes sense, and so is a generalization.

<u>Dependence on B:</u>

Note, $\frac{1}{8\pi^2}\mathrm{Tr}(F'\wedge F') \in H^4(BG, \mathbb{R})$ has integral periods. Let B_1, B_2 w/ $\partial B_i = M$ and fix ω_1', ω_2'. Gluing along M, get closed mnfld $X = B_1 \sqcup_M \overline{B}_2$ and hence

$$S(A_1) - S(A_2) = \frac{k}{8\pi^2}\int_X \mathrm{Tr}(F'\wedge F') \in \mathbb{Z}$$

So, we redefine the CS action as

$$S(A) = \frac{k}{8\pi^2}\int_B \mathrm{Tr}(F'\wedge F') \pmod 1.$$

Note: Since on overlaps, F'_α transforms in the adjoint rep of G, $\mathrm{Tr}(F'\wedge F')$ can be glued to a global 4-form on B even when $E' \longrightarrow B$ nontrivial.

<u>Observations</u>: When $E \to M$ not trivial, cannot always extend E to a bundle E' over B.

How do we take care of this most general case?

To start, note that the action is simply concerned w/ integrating the diff. form.
$\Omega(F') = \frac{k}{8\pi^2} Tr(F' \wedge F')$ over some 4-mnfld B.
So, let's generalize a bit by letting B b be a smooth singular 4-chain, since can still integrate $\Omega(F')$ over b

So, we're looking for a 4-chain b w/ bundle E' that restricts to E at $\partial b = M$.
That is, looking for 4-chain in BG that bounds $\gamma(M)$ ($\gamma: M \to BG$ classifying map).

If we have such a chain, restricting EG to this chain gives us E', after pulling it back via $\gamma' : b \to BG$

So, we're looking for 4-chain in BG with boundary M. The obstruction to the existence of such a 4-chain is measured exactly by the image $\gamma_*[M]$ in $H_3(BG, \mathbb{Z})$.

<u>Fact</u>: $H_{odd}(BG, \mathbb{Z})$ consists only of torsion.

Borel: $H^{odd}(BG, \mathbb{R}) = 0$ for compact G

So, $$0 \to \underbrace{\mathrm{Ext}^1_{\mathbb{Z}}(H_{k-1}(BG,\mathbb{Z}),\mathbb{R})}_{=0,\ \mathbb{R} \text{ divisible group}} \to \underbrace{H^k(BG,\mathbb{R})}_{=0,\ k \text{ odd}} \to \mathrm{Hom}(H_k(BG,\mathbb{Z}),\mathbb{R}) \to 0$$

$$\Rightarrow \mathrm{Hom}(H_k(BG,\mathbb{Z}),\mathbb{R}) = 0$$

$$\Rightarrow H_k(BG,\mathbb{Z}) \text{ torsion}$$

Since G compact, $H_{odd}(BG, \mathbb{Z})$ torsion $\Rightarrow$ $\exists\, m \in \mathbb{N}$ s.t. $m \cdot \gamma_*[M] = 0$ $\Rightarrow$ the bundle $G \hookrightarrow E \to M$ can be extended to the bundle $G \hookrightarrow E' \to b$ whose boundary consists of m copies of M with $E'|_{\text{any boundary component}} = E$. So, picking ω' on E' which reduces to ω on $E'|_{\partial b}$, we can define

$$m \cdot S(A) = \frac{k}{8\pi^2} \int_b \mathrm{Tr}(F' \wedge F') \pmod 1.$$

<u>Observations</u>: We've def$^{\underline{nd}}$ the general CS action for general G-bundles (G compact), but only mod $1/n$

We want to define $S(A)$ modulo 1 since shifting the action by an integer doesn't change $e^{2\pi i S(A)}$ but it might if you ~~switch~~ shift by say $1/2$ it will change the physics. So, need to resolve this ambiguity.

We know $m \cdot S(A) = \frac{k}{8\pi^2}\int_b \mathrm{Tr}(F' \wedge F') \pmod 1$

$$\iff S(A) = \frac{1}{m}\frac{k}{8\pi^2}\int_b \mathrm{Tr}(F'\wedge F') + \frac{q(b,E')}{m}$$

So, need to find $q(b,E') \in \mathbb{Z}$ for the data given.

First guess: $q(b,E') = 0$

This is wrong b/c now $S(A)$ is not independent of choice of b.

want $\frac{1}{m}\frac{k}{8\pi^2}\int_b \mathrm{Tr}(F'\wedge F') \in \mathbb{Z}$, but it's only in $\frac{1}{m}\mathbb{Z}$

So, how do we find what $q(b, E')$ should be?

Well, $\Omega(F') = \frac{k}{8\pi^2} \mathrm{Tr}(F' \wedge F') \in H^4(BG, \mathbb{R})$ has integral periods $\Rightarrow$ it lies in the image of the natural map $\rho : H^k(BG, \mathbb{Z}) \longrightarrow H^k(BG, \mathbb{R})$.

Thus, $\exists$ $\bar{\omega} \in H^k(BG, \mathbb{Z})$ s.t. $\rho(\bar{\omega}) = \Omega$ ($\bar{\omega}$ unique up to torsion). This is what we need.

Let ω be $\mathbb{Z}$-valued cocycle representing $\bar{\omega}$, then if we set $q(b, E') = \langle \gamma^*(\omega), b \rangle$, $\gamma : M \longrightarrow BG$, we get a well-def$^{\underline{nd}}$ action for arbitrary G-bundles

$$S(A) = \frac{1}{m}\left(\int_b \Omega(F') - \langle \gamma^*(\omega), b \rangle \right) \pmod 1.$$

can check everything works. In particular, $\langle \gamma^*(\omega), b \rangle \in \mathbb{Z}$ as desired and for closed 4-mnflds B

$$\int_B \Omega(F') = \langle \gamma^*(\omega), B \rangle$$

$\Rightarrow$ S independent of bounding mnfld. We can also show S only depends on $\overline{\omega}$, not choice of cocycle ω.

So, $$S = \frac{1}{m}\left(\frac{k}{8\pi^2}\int_b \mathrm{tr}(F' \wedge F') - \langle \gamma^*(\omega), b \rangle\right) \pmod 1$$

is the topological action for a connection on a bundle of order n.

Recap: If G compact, E nontrivial, the obstruction to defining $S(A) = \int_b \Omega(F')$ is given by the torsion class $\gamma_*[M] \in H_3(BG, \mathbb{Z})$. If this class has order n, then $S(A)$ can be def$\underline{\text{nd}}$ as

$$S(A) = \frac{1}{n}\left(\int_b \Omega(F') - \langle \gamma^* \omega, b \rangle\right) \pmod 1,$$

with $\omega \in H^4(BG, \mathbb{Z})$. So, the torsion info in $H^4(BG, \mathbb{Z})$ fixes this ambiguity $\Rightarrow$ the CS actions (for A) are classified by $H^4(BG, \mathbb{Z})$.

WZW Terms

Let Σ be a Riemann surface, G compact and suppose given map $g : \Sigma \longrightarrow G$, want to def$\underline{n}$ the WZW term $S(g)$.

If G simply connected, $g \overset{\text{homotopic}}{\sim}$ trivial map and extends to $g : W \longrightarrow G$, $\partial W = \Sigma$. Then can def$\underline{n}$

$$S(g) = \frac{k}{24\pi^2} \int_W \mathrm{Tr}\big(\underbrace{g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg}_{\equiv (g^{-1}dg)^3}\big).$$

$\Phi = \frac{k}{24\pi^2} \mathrm{Tr}(g^{-1}dg)^3 \in H^3(G, \mathbb{R})$ and has integral periods (i.e., lies in image $e : H^3(G,\mathbb{Z}) \longrightarrow H^3(G,\mathbb{R})$).

If G not simply connected, obstruction to extending Σ to W and g over W lies in $H_2(G, \mathbb{Z})$.

So, if G semi-simple, $H^2(G, \mathbb{R}) = 0$, $H_2(G, \mathbb{Z})$ torsion and obstruction is the torsion class $g_*[\Sigma] \in H_2(G, \mathbb{Z})$. If $g_*[\Sigma]$ has order n, then $S(g)$ can be def$^{\underline{nd}}$ as

$$S(g) = \frac{1}{n}\left(\int_W \Phi - \langle g^*\eta, W\rangle\right) \pmod 1,$$

$\partial W = n \cdot \Sigma$, $\eta \in H^3(G, \mathbb{Z})$ s.t. $e(\eta) = \Phi$.

So, once again, the torsion info in $H^3(G, \mathbb{Z})$ suffices to fix the ambiguity in the def$^{\underline{n}}$ of the WZW terms $\Rightarrow$ WZW terms classified by $H^3(G, \mathbb{Z})$.

If G not s.s. have to use diff. characters (but still reach same conclusion about classification.

Natural Map $H^4(BG,\mathbb{Z}) \longrightarrow H^3(G,\mathbb{Z})$

We know CS actions are classified by $H^4(BG,\mathbb{Z})$ and WZW terms by $H^3(G,\mathbb{Z})$ and we know that these two theories are related (gauge transforms on boundary). Hence, the correspondence between them must involve a natural map $H^4(BG,\mathbb{Z}) \longrightarrow H^3(G,\mathbb{Z})$.

We already have a candidate for the map:

consider
$$\begin{array}{ccc} G & \hookrightarrow & EG \\ & & \downarrow \pi \\ & & BG \end{array}$$
and let $\omega \in H^k(BG,\mathbb{Z})$.

Since EG contractible, $\pi^*\omega$ is exact $\Rightarrow \exists\ \beta \in H^{k-1}(EG,\mathbb{Z})$ s.t. $\pi^*\omega = \delta\beta$. Now, define $\tau(\omega)$ as the restriction of β to the fibre of EG, G. Since restricting $\pi^*(\omega)$ to fibre vanishes $\Rightarrow \tau(\omega)$ closed and further can show cohom. class of $\tau(\omega)$ doesn't depend on choice of cocycle...

get a map $\tau : H^k(BG, \mathbb{Z}) \longrightarrow H^{k-1}(G, \mathbb{Z})$.

Dijkgraaf and Witten show that deriving the WZW action from the CS action by integrating over A_0 is τ, written out in terms of diff. forms.
$\hookrightarrow$ time component

Note: $\tau : H^4(BG, \mathbb{Z}) \longrightarrow H^3(G, \mathbb{Z})$ is not necessarily onto. Hence not all group manifold models descend from a 3-dim CS theory (they show that only the group mnfld models that allow a description in terms of an extended chiral algebra will be generated by 3-dim CS theory).