Chern-Simons 10/10/11

- send John an email letting him know if he will be talking next week

- Let $G = SU(2)$

$$\begin{array}{c} P \\ \downarrow \\ M^3 \end{array} \quad \text{w/ connection } A, \quad F = dA + A \wedge A$$

can expand $\det\left(I + \frac{iF}{2\pi}\right) = c_0 + c_1 + c_2 + \cdots$

where $c_2 = \frac{1}{8\pi^2} \operatorname{Tr}(F \wedge F)$

Want 3-form c_3 such that $dc_3 = c_2$

We propose $c_3 = A \wedge dA + \frac{2}{3} A \wedge A \wedge A$. Then can check

$$dc_3 = \operatorname{Tr}\left[F^2 - A^2F - FA^2 + A^4 + \tfrac{2}{3}\left(FA^2 - AFA + A^2F - A^4\right)\right]$$

$$= \operatorname{Tr}(F^2)$$

Need to know how G acts on the CS form c_3.
Under $\varphi : P \to P$, get $g_\varphi : P \to G$ and can pull down to M using ~~Let~~ $\cancel{A \mapsto Ad_{g^{-1}}(A) + \varphi^*(\theta)}$
section. We'll see that $A \mapsto Ad_{g^{-1}}(A) + \varphi^*(\theta)$

↳ M.C. form

This tells us:

$$\varphi^* C_3 = C_3 + d\left(Ad_{g^{-1}}(A) \wedge \varphi^*(\theta) \right) - \frac{1}{3} Tr\left(\varphi^*\theta \wedge \varphi^*\theta \wedge \varphi^*\theta \right)$$

for closed M $\Rightarrow$

$$\int_M \varphi^* C_3 - \int_M C_3 = \underbrace{\frac{1}{3} \int_M Tr(\varphi^*\theta^3)}_{\in \mathbb{Z}}$$

$\Rightarrow$ $e^{2\pi i \int C_3}$ is well-def$^{\underline{nd}}$

Def$^{\underline{n}}$ the CS action as $\int_M p^* C_3(A) = S(p, A)$ (section)

Now, if $M = M_1 \sqcup M_2$ w/ A_1 and A_2 then

$$S(p,A) = S(p_1, A_1) + S(p_2, A_2)$$

Can drop p dependence since any other section is related via a gauge transformation.

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consider

$$\begin{array}{c} P \\ \downarrow \\ W = [0,t] \times M \end{array}$$

A_t, where A_t restricts to A on boundary

$$S(A_t) - S(A_0) = S_{\partial W}(A_t) = \int_0^t \int_M (F_t + ds \wedge \dot{A}_s)^2$$

$$= 2\int_0^t ds \int_M F_s \wedge \dot{A}_s$$

Want extrema: so need $F_s \equiv 0$

So, we want moduli space of flat connections

We can show, this space is

$$\mathrm{Hom}(\pi_1(M), G) / G$$

or, reducing to surfaces Y^2,

$$\begin{array}{c} Q \\ \downarrow \\ Y^2 \end{array}$$

Flat connection on Y^2 gives flat connection on $Y^2 \times [0,\infty)$

Let A_Q = space all connections on Q, then $T_A A_Q \cong \Omega^1(Y, \mathfrak{g})$

Def$^{\underline{n}}$ $\omega(a,b) = \int_Y \mathrm{Tr}\, a \wedge b$, gives symplectic form on A_Q

The space A_Q is too large – need to perform symplectic reduction.

def$\underline{n}$ $\mu : A_Q \longrightarrow (\text{Lie } G_Q)^*$

$$\shortparallel$$

$$\Omega^2(Y, \mathfrak{g})$$

We already know of a way to go from $A_Q \to \Omega^2(Y,\mathfrak{g})$, the curvature.

$$\langle d\mu|_A(a)\ \varepsilon \rangle = i_{X_\varepsilon}\omega(a)$$

$$\int_Y \varepsilon\, d_A a \qquad\qquad \int_Y d_A \varepsilon\, a$$

The space we want is $\mu^{-1}(0)/G$

Let's see how much smaller this got.

$$0 \to \Omega^0(Y,\mathfrak{g}) \xrightarrow{d_A} \Omega^1(Y,\mathfrak{g}) \xrightarrow{d_A} \Omega^2(Y,\mathfrak{g})$$

$$T_A \mathcal{M}_Q \simeq H^1(\Omega^1(Y, \mathfrak{g}))$$

moduli space of flat connections

$$\underbrace{\dim H^0}_{\text{equal}} - \dim H^1 + \dim H^2 = \dim G\, \chi(Y)$$

$$\Rightarrow \quad \dim H^1 = -\dim G\, \chi(Y) + 2 \dim H^0$$

~~Again consider~~

Let I be a complex structure on Y, then can divide:

$$\Omega^1(Y, \mathfrak{g}_{\mathbb{C}}) = \Omega^{0,1} \oplus \Omega^{1,0}$$

$$d_A = \partial_A + \bar{\partial}_A$$

Let A_0 be fixed, then

$$\bar{\partial}_A = \bar{\partial}_0 + \text{something in } \Omega^{0,1}$$

so can quotient by $\mathrm{Aut}(\mathfrak{g} \otimes \mathbb{C})$ giving another moduli space M_I, and can prove $M_I \simeq M_Q$