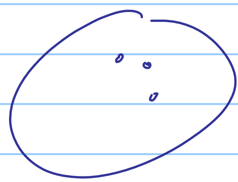
Axiomatic TFT & Chern-Simons theory

Note Title 10/25/2011

1. Hilbert space in genus 0: Comes from mathematical CFT



points in rep.s R_i

large k limit $\rightsquigarrow$ uncoupled particles

$\mathcal{H} = \mathrm{Inv}(\bigotimes_i R_i)$ ideas: 1) $\mathrm{Inv}(-)$ is a conservation of charge $= 0$

2) $\mathcal{H} = 0$ unless all R_i are integrable rep.s of the loop group at level k

marked points

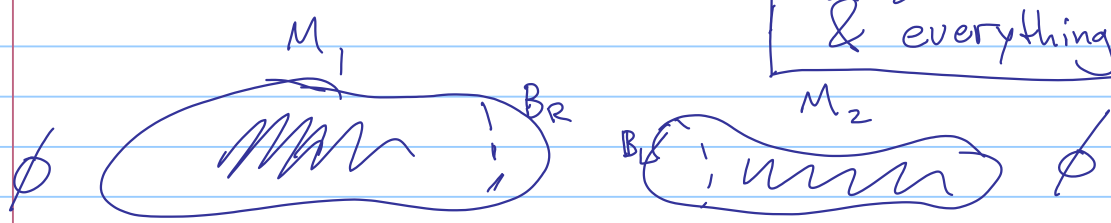
0, 1, 2) easy

3) $\dim = N_{ijk}$ $\rightsquigarrow$ Verlinde formulas

>3) verlinde formulas

Special case $\mathrm{Inv}(R \otimes R \otimes R \otimes R)$ determined by Isotypical decomp $R \otimes R = \sum_{i=1}^{s} E_i$

Simplest definition of 3d TFT: $\otimes$-Functor $\mathrm{Bord}_3 \xrightarrow{Z} \mathrm{Vect}$, where are 2-manifolds & 3-mflds come decorated w/ embedded curves, & everything is framed.



Idea 1: 3-mfld bounding Σ gives vector in $F(\Sigma)$, or a dual vector in $F(\Sigma^{op})$

Idea 2: property of <u>1-dim'l</u> vector spaces

$$\langle x, v_{B_R}\rangle\langle v_{B_L}, \psi\rangle = \langle x, \psi\rangle\langle v_{B_R}, v_{B_L}\rangle$$

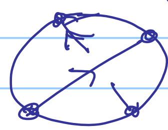
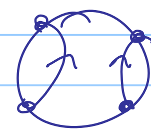
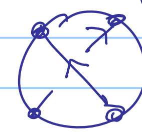
i.e. $Z(M_1)Z(M_2) = Z(S^3)\,Z(M_1 \# M_2)$ ⟵ works with decoration

If $L_1, \dots, L_s$ separable links in S^3, then this implies

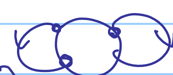
$$\frac{Z(S^3; L_1, \dots, L_s)}{Z(S^3)} = \prod_i \frac{Z(S^3; L_i)}{Z(S^3)}$$

$\boxed{\text{Skein relations:}}$ $G = SU(N)$, $R =$ defining rep,
$R \otimes R = \mathrm{Sym}^2(V) \oplus \Lambda^2(V) \rightsquigarrow$ so $\dim \mathcal{H} = 2$

imagine L in S^3 and choose a region with a single crossing, cut out a ball containing the crossing $\rightsquigarrow$ look a planar projections

$\dim \mathcal{H} = 2 \implies \alpha$  $+ \beta$  $+ \gamma$  $= 0$

This relation suffices to compute $Z(L)$ from the unknot (although you have to check that these relations are consistent)

Capping these pictures off with  gives

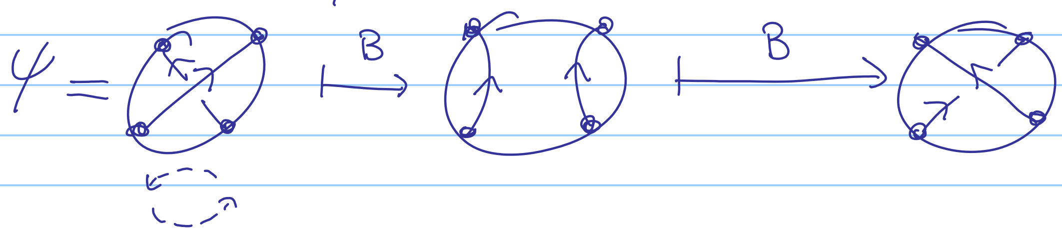
$$\langle \text{unknot} \rangle \stackrel{\text{def}}{=} \frac{Z(S^3; \text{unknot})}{Z(S^3)} = \frac{-(\alpha+\gamma)}{\beta}$$

$\boxed{\text{Key idea:}}$ actions of diffeomorphisms of surface on spaces of conformal blocks $\longrightarrow$ $\boxed{\text{Mention mapping cylinders}}$

1) <u>Dehn twist:</u> diffeomorphism of fixing all points. Has the effect of rotating the framing around a single point

$\longrightarrow$ acts on $\mathcal{H}_R$ by multiplication by $e^{2\pi i h_R}$ where h_R is the "conformal weight of the primary field in the R representation"

2) <u>Half-monodromy:</u>



again $\dim \mathcal{H} = 2$, so

$$B^2 - \mathrm{tr}(B)B + \det(B) = 0$$

$$\alpha = \det(B),\ \beta = -\mathrm{tr}(B),\ \gamma = 1$$

From CFT, $\lambda_i = \pm \exp(i\pi(2h_R - h_{E_i}))$ where $\pm$ is parity of E_i under $R \otimes R$

In our case $\lambda_1 = \exp\left(\frac{i\pi(-N+1)}{N(N+k)}\right)$, $\lambda_2 = -\exp\left(\frac{i\pi(N+1)}{N(N+k)}\right)$

(see p. 382 for α, β, γ) Renormalizing and introducing $q = \exp\left(\frac{2\pi i}{N+k}\right)$ the relation becomes

$$-q^{N/2} L_+ + (q^{+1/2} - q^{-1/2}) + q^{-N/2} L_- = 0$$ for links in S^3 with standard (unlinking) framing

$$\langle C \rangle = \frac{q^{N/2} - q^{-N/2}}{q^{1/2} - q^{-1/2}} = [n]_q \longrightarrow N, \text{ i.e. } \mathrm{tr}(\mathrm{ID}) \text{ as } k \to \infty$$

Enter Surgery: A diffeomorphism acts on $Z(T^2)$, so if we do surgery along a curve in $M \mapsto \tilde{M}$, the invariant transforms as

$$Z(M) = \langle \chi, \psi \rangle \longmapsto Z(\tilde{M}) = \langle \chi, K\psi \rangle$$

What is $Z(T^2)$?

Construct a basis by $\boxed{\text{considering } T^2 \text{ as bndary}}$ of solid torus, inserting a Wilson line around the interior loop in the representation R_i = highest weight rep of LG at level k

choice

highest weight space is an irrep. of G

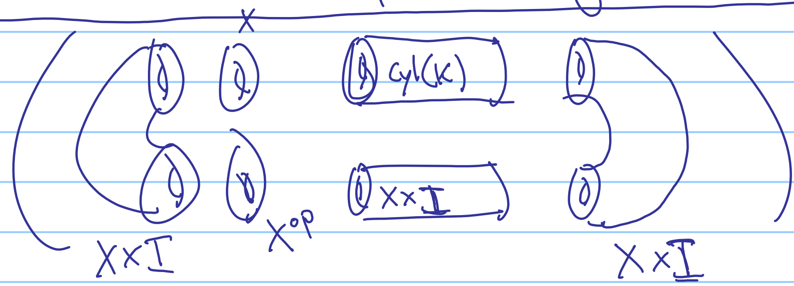
↳ this gives a vector $v_i \in Z(T^2)$

Fact: they form a basis...

If we do surgery K along a Wilson loop in rep R_i, then we get

$$Z(\tilde{M}; R_i) = \langle \chi, K v_i \rangle = K_i^j \, Z(M; R_j)$$

$\boxed{\text{Special case of path integrals on } X \times S^1:}$ K be a diffeom. of X

Z 

$= \mathrm{tr}_{\mathcal{H}_X}(K) = Z(X \times_K S^1)$

In particular, for $K = \mathrm{id}$, get $\dim \mathcal{H}_X$.

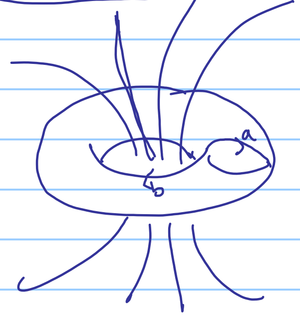
Applying this to $S^1 \times S^2$ with Wilson lines the S^1-orbits this gives

$$Z(S^1 \times S^2; \{R_i\}) = \dim \mathcal{H}(S^2 \text{ with charges in } R_i)$$

$$= \dim \mathrm{Inv}\left(\bigotimes_i R_i\right) \quad \text{for large } k$$

Chern-Simons on S^3:

S^3 obtained by surgery on $S^2 \times S^1 = (D^2 \times S^1) \times_{T^2} (D^2 \times S^1)$



the surgery corresponds to the S matrix

$a \longmapsto b$

$b \longmapsto -a$

From CFT, can compute the action of the S-matrix

e.g. $G = SU(2)$, spin $n/2$ for $n = 0, \dots, k$ $\quad S_{m,n} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{(m+1)(n+1)\pi}{k+2}\right)$

Adding a Wilson line parallel to the line on which we're doing the surgery on $S^1 \times S^2$ to obtain S^3, we get

$$Z(S^3; R_j) = \sum_i S_0^i \, Z(S^2 \times S^1; R_i, R_j) = S_{0,j}$$

$$Z(S^3; R_i, R_j) = \sum_i S_0^i \, Z(S^2 \times S^1; R_i, R_j, R_k) = \sum_i S_0^i N_{ijk}$$

But we already had a way of describing $Z(S^3, \{R_\alpha\})$ from the characteristic polynomial of B on $\mathcal{H}_{S^2, 4 \text{ marked points}}$

$$\Rightarrow \langle C \rangle = \frac{Z(S^3; R_1)}{Z(S^3; R_0)} = \frac{S_{0,1}}{S_{0,0}} = \frac{\sin(2\pi/(k+2))}{\sin(\pi/(k+2))}$$

$$= [N]_q = \frac{q^{N/2} - q^{-N/2}}{q^{1/2} - q^{-1/2}}$$

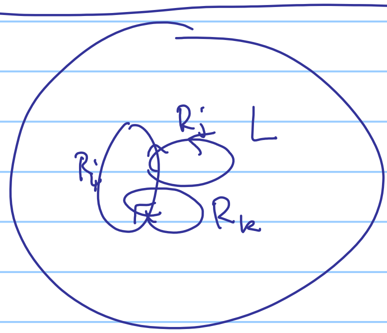
Also for two unlinked wilson loops:

$$\frac{S_{0,ij} S_{0,ik}}{S_{0,0}} = \sum S_0^i N_{ijk}$$

General idea is to compare with the first surgery formula

$$Z(M_1 + M_2; L_1 + L_2) \cdot Z(S^3; C) = Z(M_1; L_1) \cdot Z(M_2; L_2)$$

$\boxed{\text{General Verlinde formula:}}$



$$Z(S^3; L) = \sum_i S_i^m \, Z(S^2 \times S^1; R_m, R_j, R_k) = \sum S_i^m N_{jkm}$$

In general the surgery from $S^2 \times S^1 \rightsquigarrow S^3$ will create links of this form.

Using the fact that $\mathcal{H}_{S^2, R \otimes \bar{R}}$ is 1-dim'l we can split the L, and we get a formula

$$Z(S^3; L) \cdot Z(S^3; R_i) = \underbrace{Z(S^3; L(R_i; R_j))}_{= S_{ij}} \, Z(S^3; L(R_i, R_k))$$

This gives: $\dfrac{S_{ij} \cdot S_{ik}}{S_{0i}} = \sum S_i^m N_{jkm}$ ← the Verlinde formula!