Three Dimensional Topological Field Theory

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Abstract

The purpose of this paper is to outline the ideas used by Dijkgraaf and Witten to classify topological actions defined on arbitrary 3-manifolds and compact Lie groups [3]. We intend to provide the details which were overlooked by the authors, while simply stating the results which were given adequate explanation in the original paper. Once we have constructed the general definition for an action on an arbitrary 3-manifold, we then restrict ourselves to the case of finite gauge group. It is here where we carry out explicit calculations.

1 Topological Actions

Chern-Simons theory is a gauge theory*. Hence, it should have a mathematical description in terms of fibre bundles and connections. Thus, let M be a three-dimensional oriented manifold, G a compact Lie group, E a principal G-bundle over M, and ω a connection form on E. When E is trivial, the connection ω on E can be represented by a $\mathfrak{g}$-valued one-form on M, which we denote by A (in particular, $A = s^*(\omega)$ where $s : M \to E$ is a global section). In this case, it makes sense to integrate A over M. So, we have the following definition.

Definition 1.1. *For E trivial, we define the Chern-Simons action functional as*

$$S(A) = \frac{k}{8\pi^2} \int_M Tr\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right), \tag{1}$$

where Tr is an ad_G-invariant symmetric bilinear form on $\mathfrak{g}$ and k is a constant.

We now show that the parameter k must be an integer if the quantum measure $e^{2\pi i S}$ is to be gauge invariant. For trivial bundles, gauge transformations $\phi \in \mathrm{Aut}(E_x)$‡ are in a $1-1$ correspondence with mappings $g_\phi : M \to G$. Furthermore, by direct calculation, one can show that the action (1) is invariant under the component of the gauge group that contains the identity. So, let $g_0 : M \to G$ be a trivial gauge transformation and let g_1 be any other gauge transformation. Let's now try to construct a homotopy between g_0 and g_1. The obstruction to such a homotopy is an element in $H^{n+1}\big(M \times [0,1], M \times \{0,1\}, \pi_n(G)\big)$ which, by the suspension theorem, is isomorphic to $H^n\big(M, \pi_n(G)\big)$. For compact simple Lie groups G, we have that $\pi_3(G) \cong \mathbb{Z}$. Hence, it is not always possible to define a homotopy between g_0 and g_1. Consequently, there exists non-trivial gauge transformations. Gauge transformations associated with non-zero elements of $\pi_3(G)$ are called gauge transformations of non-zero "winding number." Under a gauge

*The uninformed reader is directed to the book by Rider "Quantum Field Theory" and/or the second volume of Weinberg "Quantum Field Theory."

‡Here by E_x we mean the fibre over the point $x \in M$. That is, gauge transformations are mathematically defined as fibre-wise bundle automorphisms.

transformation of winding number m, the transformation law of (1) is (assuming M is closed †)

$$S(A) \mapsto S(A) + km.$$

Thus, we conclude, if k is an integer then the quantum measure $e^{2\pi i S}$ is invariant under gauge transformations.

Let's now investigate the validity of definition 1.1 for the general case. When G is a connected, simply connected, compact Lie group one has that $\pi_i(G) = 0$ for $i = 0, 1, 2$. Thus, in this case, any G-bundle over a three-dimensional manifold M is necessarily trivial (and definition 1.1 can serve as the general definition for the Chern-Simons action). Indeed, one can define a section s over the zero cell M^0 by arbitrarily assigning points in the corresponding fibre. To extend s to M^1 one must lift edges in M^1 to E. Hence, using a trivialization, this is equivalent to the connectivity of G. Therefore, since $\pi_0(G) = 0$, we can extend over M^1. Similarly, since $\pi_1(G) = \pi_2(G) = 0$, we can extend s over M^2 and $M^3 = M$. That is, we can define a global section $s : M \to E$; equivalently, E is trivial over M. In general, for arbitrary G, it's not the case that $\pi_i(G) = 0$ for $i = 0, 1, 2$ and so, it's not true that one can always trivialize E. When E is not trivial it no longer makes sense to represent the connection ω by a $\mathfrak{g}$-valued one-form on M (when E is not trivial there does not exist a global section $s : M \to E$ to pull ω down to M). Therefore, the previous expression for $S(A)$ does not hold for non-trivial bundles. So, we need to give a more general expression for $S(A)$, one that holds even when E is non-trivial.

In order to gain a better understanding of the situation let's assume that we have a trivial bundle E over an oriented three-manifold M. Now, it's a well-known result from cobordism theory that every three-manifold M bounds a four-manifold B. Further, one can always extend a trivial bundle E over M to a bundle E' over B which reduces to E when restricted to $\partial B = M$. In addition, one can pick a connection ω' on E' which reduces to ω on $E'|_{\partial B}$ (just use a partition of unity to extend ω on $E = E'|_{\partial B}$ to ω' on E'). Now, by Stokes' theorem and by the fact that $d\big(\mathrm{Tr}\,(A \wedge dA + 2/3 A \wedge A \wedge A)\big) = \mathrm{Tr}(F \wedge F)$ (where F is the curvature of A), we can rewrite (1) as‡

$$S(A) = \frac{k}{8\pi^2} \int_B \mathrm{Tr}(F' \wedge F').$$

Let us now see how this expression depends on the choice of bounding manifold B and extension ω'. Suppose we have two choices B_1, B_2 with $\partial B_1 = M = \partial B_2$, along with two extensions ω'_1 and ω'_2. Then, gluing B_1 and B_2 along M, gives a closed manifold $X = B_1 \sqcup_M \bar{B}_2$ (here $\bar{B}_2$ denotes B_2 with the opposite orientation). Thus, forming a connection ω' which interpolates ω'_1 and ω'_2, we have

$$S(A_1) - S(A_2) = \frac{k}{8\pi^2} \int_X \mathrm{Tr}(F' \wedge F').$$

Now, the right-hand side is an integer since the Chern-Weil 4-form $\frac{1}{8\pi^2}\mathrm{Tr}(F' \wedge F') \in H^4(BG, \mathbb{R})$ has integral periods (i.e., pairing $\frac{1}{8\pi^2}\mathrm{Tr}(F' \wedge F')$ with a closed four-manifold X gives an integer; that is $\frac{1}{8\pi^2}\int_X \mathrm{Tr}(F' \wedge F') \in \mathbb{Z}$). Therefore, we are lead to a slightly more general definition of the Chern-Simons action.

Definition 1.2. *Let E' be a bundle over a four-manifold B, where $\partial B = M$, let A' be a connection form which reduces*

†In general, for a gauge transformation g of winding number m,

$$S(A) \mapsto S(A) - \frac{k}{8\pi^2} \int_M d\big(\mathrm{Tr}(gA \wedge dg^{-1})\big) + \frac{k}{24\pi^2} \int_M \mathrm{Tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg).$$

And so, if $\partial M = \emptyset$ the second term vanishes, while for the WZW term we have $\frac{1}{24\pi^2}\int_M \mathrm{Tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg) = m$.

‡Note, since on overlaps $U_i \cap U_j$ of B the curvature F transforms in the adjoint representation of G, $F_j = ad(g^{-1}) \cdot F_i$, and since Tr is an ad_G-invariant form, the expression $\mathrm{Tr}(F \wedge F)$ can be patched together to yield a globally defined $\mathfrak{g}$-valued 4-form on B which can be integrated.

to A at the boundary and denote its associated curvature by F'. We define the Chern-Simons action as

$$S(A) = \frac{k}{8\pi^2}\int_B Tr(F' \wedge F') \qquad (mod\ 1). \tag{2}$$

This expression reduces to (1) when (1) makes sense, and so does represent a more general definition of the Chern-Simons functions. However, if E is not trivial then it's not possible, in general, to extend E to E'. And so, (2) is not the most general definition of the Chern-Simons action.

If E is not topologically a product $G \times M$, (2) needs to be modified as it is not always possible to extend E to a bundle E' over a 4-manifold B which bounds M. From the (so-far) definition of the Chern-Simons action, it appears that we are only concerned with integrating a differential form $\Omega(F) = \frac{k}{8\pi^2}\mathrm{Tr}(F \wedge F)$ over some 4-manifold B. So, instead of considering extending the bundle E to a bundle over a bounding four-manifold B, let's be more general and allow for B to be a singular 4-chain, since a differential 4-form can be integrated over any such 4-chain. Note, since we are looking for a singular 4-chain B and bundle E' which restricts to E at $\partial B = M$, we are actually trying to find a 4-chain in the classifying space BG that bounds $\gamma(M) \in BG$ (here $\gamma : M \longrightarrow BG$ is the classifying map). Restricting the universal bundle EG to this chain B would then give us our desired bundle E', after pulling it back from the classifying space. So, we are looking for a singular 4-chain in BG which has boundary M. The obstruction to such a chain is then given by the image $\gamma_*[M]$ in $H_3(BG, \mathbb{Z})$; that is, if $\gamma_*[M]$ vanishes then, by definition of singular homology, there must exist some 4-chain B whose boundary is M. So, we now must investigate if and when this obstruction vanishes.

Theorem 1.1 (Borel)**.** *Let G be a compact Lie group, then all odd real cohomology vanishes,*

$$H^{odd}(BG, \mathbb{R}) = 0.$$

Further, if G is finite then all real cohomology vanishes, $H^*(BG, \mathbb{R}) = 0$.

Proof. We give a sketch of the proof for the first statement, leaving the second statement to the reader (see for e.g. [1]). Let G be a compact Lie group and let T be a toral subgroup of G. Thus, we have an inclusion $\iota : T \hookrightarrow G$ which, as can be shown, induces an inclusion in cohomology $\iota^* : H^*(BG, \mathbb{R}) \hookrightarrow H^*(BT, \mathbb{R})$, for all n. Now, since T is toral, we have that T is basically given by n copies of S^1, $T \cong S^1 \times \cdots \times S^1$. So, we have

$$\begin{aligned} H^*(BT, \mathbb{R}) &\cong H^*\big(BU(1) \times \cdots \times BU(1), \mathbb{R}\big) \\ &\cong \mathbb{R}[x_1, ..., x_n], \end{aligned}$$

where the x_k's have even degree. This implies $H^{odd}(BT, \mathbb{R}) = 0$ - since any polynomial in x_k's necessarily has an even degree. And so, for compact G, $H^{odd}(BG, \mathbb{R}) = 0$. ☐

Corollary 1.1. *If G is a compact group then its odd cohomology and homology consists completely of torsion, while if G is a finite group then its cohomology and homology is completely torsion.*

Proof. The proof that the cohomology consists completely of torsion follows immediately from the previous theorem along with the fact that torsion classes are elements of the kernel of the natural map $H^k(BG, \mathbb{Z}) \longrightarrow H^k(BG, \mathbb{R})$. Next, let k be an odd integer and consider the universal coefficient theorem for cohomology,

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}\big(H_{k-1}(BG, \mathbb{Z}), \mathbb{R}\big) \longrightarrow H^k(BG, \mathbb{R}) \longrightarrow \mathrm{Hom}\big(H_k(BG, \mathbb{Z}), \mathbb{R}\big) \longrightarrow 0.$$

Since $\mathbb{R}$ is divisible $\mathrm{Ext}^1_{\mathbb{Z}}\big(H_*(BG, \mathbb{Z}), \mathbb{R}\big) = 0$ and, since k is odd, $H^k(BG, \mathbb{R}) = 0$ (by the previous theorem). Hence, we have that $\text{Hom}\big(H_k(BG,\mathbb{Z}),\mathbb{R}\big) = 0$; that is, $H_k(BG,\mathbb{Z})$ consists completely of torsion when k is odd. The finite case is analogous. $\square$

If G is compact then the real odd homology $H_{odd}(BG,\mathbb{Z})$ consists completely of torsion. Thus, for every class $[c] \in H_{odd}(BG,\mathbb{Z})$ there exists some positive integer m such that $m \cdot [c] = 0$. In particular, we have

$$m \cdot \gamma_*[M] = 0,$$

for some $m \in \mathbb{N}$. That is, any general G-bundle over a 3-manifold M can be extended to a G-bundle E' over a 4-chain B, whose boundary consists of m copies of M, such that the restriction of E' on all boundary components is isomorphic to E. Picking a connection ω on E' which reduces to the connection ω_0 on $E'_{\partial B}$, we can always define

$$m \cdot S = \frac{k}{8\pi^2} \int_B \text{Tr}(F \wedge F) \qquad (\text{mod } 1).$$

So, what is left in defining the Chern-Simons action for a 3-manifold M is to resolve this m-fold ambiguity in a way that is consistent with the properties of a topological field theory - factorization and unitarity.

To begin, note that $\Omega(F) = \frac{k}{8\pi^2}\text{Tr}(F \wedge F)$ represents some real class $[\Omega] \in H^4(BG,\mathbb{R})$. Further, $\Omega(F)$ has integral periods, implying that $[\Omega]$ lies in the image of the natural map $\rho : H^4(BG,\mathbb{Z}) \longrightarrow H^4(BG,\mathbb{R})$. Hence, there exists some integral class $[\beta] \in H^4(BG,\mathbb{Z})$ such that $\rho([\beta]) = [\Omega]$. Now, since Tor $H^4(BG,\mathbb{Z}) \subset \ker(\rho)$, the choice of particular $[\beta]$ such that $\rho([\beta]) = [\Omega]$ is unique only up to torsion. Indeed, let $[\beta']$ be a torsion element in $H^4(BG,\mathbb{Z})$ then $\rho([\omega] + [\beta']) = \rho([\beta]) + \rho([\beta']) = \rho([\beta]) = [\Omega]$. It is precisely the choice of which $[\beta]$ is used to define $[\Omega]$ that alleviates the ambiguity above. In particular, we have the following definition.

Definition 1.3 (General Case). *Let β be any integer-valued cochain representing the class $[\beta] \in H^4(BG,\mathbb{Z})$. Then, we define the topological action for a connection no a bundle of order m to be*

$$S = \frac{1}{m}\left(\int_B \Omega(F) - \langle \gamma^*(\beta), B\rangle\right) \qquad (\textit{mod } 1), \tag{3}$$

where $\gamma : B \longrightarrow BG$ is the classifying map.

Let's now check that this definition is completely (well-)defined. That is, we need to check that definition (1.3) is independent of the bounding chain B, the definition depends only on the class $[\beta]$ and not the choice of cochain β representing the class and, finally, that the action is invariant under homotopy transformations of the classifying map γ. So, to proceed, note that on closed 4-chains B we have

$$\int_B \Omega(F) = \langle \gamma^*(\beta), B\rangle;$$

i.e., this tells us that (3) is independent of the bounding chain B and the way in which the connection and bundle was extended to B. Next, under the shift $\beta \longmapsto \beta + \delta\nu$, where $\nu \in C^3(BG,\mathbb{Z})$, the variation of S becomes

$$\delta S = -\frac{1}{m}\langle \gamma^*(\delta\nu), B\rangle = -\frac{1}{m}\langle \gamma^*(\nu), m \cdot M\rangle = -z,$$

where $z \in \mathbb{Z}$ since $\nu \in C^3(BG,\mathbb{Z})$; that is, $\delta S = 0$ (mod 1). Hence, the action only depends on the class $[\beta]$. Finally, the action is indepedent under homotopy transformations of the classifying map. Indeed, homotopic maps induce the same morphisms on cohomology, so if γ and γ' are homotopic then $\gamma^*(\beta) = \gamma'^*(\beta)$ in $H^4(BG,\mathbb{Z})$ and since the pairing in the action only depends on the cohomology class $[\beta]$, we have

$$\langle \gamma^*(\beta), B\rangle = \langle \gamma'^*(\beta), B\rangle.$$

Therefore, the action is (well-)defined. Note, it is also possible to show that, for closed manifolds, the action is gauge invariant. While for manifolds with boundary, under a gauge transformation the variation of S depends only on the boundary data - as is expected.

Since we'll need the result later, let's now see how the definition 1.3 depends on the torsion information in $[\beta]$. When $[\beta]$ is completely torsion, it follows that $\rho([\beta]) = 0 = [\Omega]$. Hence, in this case, the action becomes

$$S = \frac{1}{m}\langle \gamma^*(\beta), B\rangle \qquad (\text{mod } 1).$$

What is more, through the isomorphism Tor $H^4(BG,\mathbb{Z}) \cong H^3(BG,\mathbb{R}/\mathbb{Z})$ †, we know that there is a cochain $\alpha \in H^3(BG,\mathbb{R}/\mathbb{Z})$ such that $\delta\alpha = \beta$. Consequently, we can further rewrite the action

$$\begin{aligned} S &= \frac{1}{m}\langle \gamma^*(\beta), B\rangle \qquad (\text{mod } 1) \\ &= \frac{1}{m}\langle \gamma^*(\delta\alpha), B\rangle \qquad (\text{mod } 1) \\ &= \frac{1}{m}\langle \gamma^*(\alpha), m\cdot[M]\rangle = \langle \gamma^*(\alpha), [M]\rangle \in \mathbb{R}/\mathbb{Z}. \end{aligned}$$

Note, when G is finite it is subsequently true that $\Omega(F) = 0$. Hence, we are lead to the following.

Definition 1.4. *When G is finite, we define the topological action as*

$$S = \langle \gamma^*(\alpha), [M]\rangle, \tag{4}$$

where $\gamma : M \longrightarrow BG$ is the classifying map and $\alpha \in H^3(BG,\mathbb{R}/\mathbb{Z})$.

2 Quantization

Chern-Simons theory yields a topological quantum field theory (TQFT). The simplest and quickest definition of a TQFT is the following, due to Atiyah: A TQFT is a symmetric monoidal functor $Z : (\text{Cob}, \sqcup) \longrightarrow (\text{Vect}, \otimes)$ from the category of cobordisms, with monoidal structure the disjoint union $\sqcup$, to the category of vector spaces, whose monoidal structure is given by the tensor product of vector spaces $\otimes$. Thus, the quantized Chern-Simons theory associates a vector space V_Σ to surfaces and to a 3-manifold M it assigns an element of $V_{\partial M}$. ‡ Further, we assume $V_\emptyset = \mathbb{C}$. We shall now see how this all works (see [4]).

To M, a closed 3-manifold, the Chern-Simons assigns the path integral

$$M \longmapsto Z(M) = \int_{\mathcal{A}(E)/\mathcal{G}} e^{2\pi i S(A)} \mathcal{D}(A), \tag{5}$$

where $\mathcal{A}(E)/\mathcal{G}$ is the reduced phase space of the theory. From the Euler-Lagrange equations (or moment maps) one can

†Indeed, the short exact sequence $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 0$ induces a long exact sequence

$$\cdots \longrightarrow H^k(BG,\mathbb{R}) \longrightarrow H^k(BG,\mathbb{R}/\mathbb{Z}) \longrightarrow H^{k+1}(BG,\mathbb{Z}) \longrightarrow H^{k+1}(BG,\mathbb{R}) \longrightarrow \cdots.$$

Since $H^*(BG,\mathbb{R}) = 0$ when G is finite, it follows that $H^{k+1}(BG,\mathbb{Z}) \cong H^k(BG,\mathbb{R}/\mathbb{Z})$ (for all k) for finite groups G.

‡To be exact, let $M = \Sigma_1 \to \Sigma_2$ be a cobordism, then $\partial M = \Sigma_2 \sqcup \overline{\Sigma}_1$ and the quantized Chern-Simons theory should assign an element in $\text{Hom}(\Sigma_1,\Sigma_2)$ (it's a functor after all). However, we also have $\text{Hom}(\Sigma_1,\Sigma_2) \cong V_{\Sigma_2} \otimes V^*_{\Sigma_1}$. That is, the Chern-Simons theory assigns to 3-manifolds vectors in the vector space associated to its boundary components.

calculate that the reduced phase space corresponds to flat connections on $G \hookrightarrow E \to M$ modulo gauge transformations $\mathcal{G}$. To a closed surface Σ the Chern-Simons theory assigns

$$\Sigma \longmapsto V_\Sigma = L^2\big(\mathcal{A}(E')/\mathcal{G}\big), \tag{6}$$

where here the reduced phase space corresponds to flat connections on $G \hookrightarrow E' \to \Sigma$ modulo gauge transformations (we'll explicitly give the L^2 measure later for the finite G case). For 3-manifolds M with boundary the Chern-Simons theory assigns

$$M \longmapsto Z(\partial M) = \int_{\mathcal{A}(\partial E)/\mathcal{G}} e^{2\pi i S(A)} \mathcal{D}(A), \tag{7}$$

where $\mathcal{A}(\partial E)/\mathcal{G}$ is the restriction of $\mathcal{A}(E)/\mathcal{G}$ to the boundary ∂E.

Remark 2.1. *When M is closed $\partial M = \emptyset$ and so, $V_{\partial M} = V_\emptyset = \mathbb{C}$. Thus, $Z(M)$ indeed gives an element in $V_{\partial M=\emptyset}$ since $Z(M) \in \mathbb{C}$ for closed M. Additionally, it can be shown (see [4]) that $Z(\partial M)$ defines an element in $V_{\partial M}$. One can go even further and calculate the quantum invariants of 1-manifolds and points, resulting in categories and 2-categories [6].*

Calculations

Let's now make some concrete calculations. To begin, we need to gain further insight into the reduced phase space. We have the following theorem relating flat bundles and G-representations of $\pi_1(M)$.

Theorem 2.1. *Let M be any smooth connected manifold. Choose a basepoint $x \in M$. Then the correspondence which sends each flat G-bundle over M to its holonomy homomorphism induces a bijection*

$$\{\textit{isomorphism class of flat } G\textit{-bundles over } M\} \cong \textit{Hom}(\pi_1(M,x), G)/G,$$

which is independent of the basepoint x_i. Note, here G acts on $\textit{Hom}(\pi_1(M,x), G)/G$ by conjugation.

Proof. See [5]. ∎

Thus, we can now identify the reduced phase space $\mathcal{A}(E)/\mathcal{G}$ with G-representations of $\pi_1(M)$,

$$\mathcal{A}(E)/\mathcal{G} \cong \mathrm{Hom}(\pi_1(M,x), G)/G. \tag{8}$$

We'll now compute some quantum invariants of 2- and 3-manifolds in the case of finite gauge group G. In this case, the partition function reduces to a finite sum

$$\int_{\mathcal{A}(E)/\mathcal{G}} e^{2\pi i S(A)} \mathcal{D}(A) \longmapsto \frac{1}{|G|} \sum_{\gamma \in \mathrm{Hom}(\pi_1(M,x),G)} e^{2\pi i S}.$$

As we have seen, if G is finite all real cohomology $H^*(BG, \mathbb{R})$ vanishes, implying that all integral cohomology is completely torsion. Hence, under the natural map $\rho : H^k(BG, \mathbb{Z}) \longrightarrow H^k(BG, \mathbb{R})$ the class $[\beta] \in H^4(BG, \mathbb{Z})$ maps to the trivial class, $\rho([\beta]) = 0$. However, from the definition of the action, $[\Omega] = \rho([\beta]) = 0$ ‡. This tells us that the action becomes $S = \frac{1}{m}\langle \gamma^*(\beta), [B]\rangle \pmod 1$. Further, if β is torsion then it determines a 3-cocycle $\alpha \in H^3(BG, \mathbb{R}/\mathbb{Z})$ through the isomorphism $\mathrm{Tor}H^4(BG, \mathbb{Z}) \cong H^3(BG, \mathbb{R}/\mathbb{Z})$. That is, when G is finite, the action reduces to the much simpler form

$$e^{2\pi i S} = \langle \gamma^*(\alpha), [M]\rangle \in S^1,$$

‡Alternatively stated, every G-bundle with G finite is necessarily flat; i.e., $\Omega(F) = 0$.

where now we view $\alpha \in H^(BG, S^1)$ (under the isomorphism $\mathbb{R}/\mathbb{Z} \cong S^1$).

We now would like to work out the specific calculations for the $G = \mathbb{Z}_2$ Chern-Simons theory. To begin, let's find the classifying space $B\mathbb{Z}_2$. From the definition of BG, namely that it can be realized as the quotient space

$$BG = EG/G,$$

along with $E\mathbb{Z}_2 = S^\infty$, we see that the classifying space for $\mathbb{Z}_2$ takes the form

$$B\mathbb{Z}_2 = S^\infty/\mathbb{Z}_2 = S^\infty/x \sim -x.$$

Recalling that $\mathbb{R}P^\infty = S^\infty/x \sim -x$, we conclude

$$B\mathbb{Z}_2 = \mathbb{R}P^\infty.$$

The projective space $\mathbb{R}P^\infty$ is a type $K(\mathbb{Z}_1, 1)$ Eilenberg-MacLane space. Recall, for Eilenberg-MacLane spaces of type $K(\mathbb{Z}_q, n)$, we have (see exercise 18.8 page 245 of [2])

$$H^n\big(K(\mathbb{Z}_q, 1); \mathbb{Z}\big) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ \mathbb{Z}_q & \text{if } n > 0 \text{ and even,} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $H^4(B\mathbb{Z}_2; \mathbb{Z}) \cong H^3(B\mathbb{Z}_2; \mathbb{R}/\mathbb{Z}) \cong \mathbb{Z}_2$. So, since the action depends on a particular choice of cocycle $[\alpha] \in H^3(\mathbb{R}\mathbf{P}^\infty, \mathbb{R}/\mathbb{Z})$ and since $H^3(\mathbb{R}\mathbf{P}^\infty, \mathbb{R}/\mathbb{Z}) \cong \mathbb{Z}_2$, we see that there exists two cases: (1) the untwisted theories $[\alpha] = 0$, and (2) the twisted theories $[\alpha] \neq 0$.

For the untwisted case, $S = 0$ and the path integral becomes

$$Z(M) = \frac{\big|\big(\mathrm{Hom}(\pi_1(M, x), \mathbb{Z}_2)\big)\big|}{|\mathbb{Z}_2|}.$$

So, calculations in the untwisted case are rather straightforward. For instance, $Z(M) = 1/2$ for all connected simply connected and closed 3-manifolds M. In fact, we have the following proposition.

Proposition 2.1. *Let G be a finite group and let M be any closed, connected, simply connected 3-manifold. Then, the untwisted Chern-Simons partition function of M for any finite G is given by*

$$Z(M) = \frac{1}{|G|}.$$

Proof. This is follows immediately from the definition of $Z(M)$ and the fact that M is connected and simply connected. Indeed, for a closed 3-manifold M

$$Z(M) = \frac{\big|\big(\mathrm{Hom}(\pi_1(M, x), G)\big)\big|}{|G|},$$

while if M is connected and simply connected $\mathrm{Hom}(\pi_1(M, x), G) \cong \{1\}$. □

Consider now the case of $M = \mathbb{R}P^3$. The fundamental group of $\mathbb{R}P^3$ is cyclic of order two, $\pi_1(\mathbb{R}P^3) \cong \mathbb{Z}_2$.

Therefore, the untwisted $\mathbb{Z}_2$ Chern-Simons partition function of $\mathbb{R}P^3$ is given by

$$\begin{aligned} Z(\mathbb{R}P^3) &= \frac{\big|\big(\mathrm{Hom}(\pi_1(\mathbb{R}P^3), \mathbb{Z}_2)\big|}{|\mathbb{Z}_2|} \\ &= \frac{\big|\big(\mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2)\big|}{|\mathbb{Z}_2|} \\ \frac{2}{2} &= 1. \end{aligned}$$

However, the twisted $\mathbb{Z}_2$ Chern-Simons partition function vanishes on $\mathbb{R}P^3$. Indeed, there are two possible $\mathbb{Z}_2$-bundles over $\mathbb{R}P^3$. Further, the nontrivial classifying map $\gamma_1 : \mathbb{R}P^3 \longrightarrow \mathbb{R}P^\infty$ (recall, $B\mathbb{Z}_2 = \mathbb{R}P^\infty$) corresponds to the embedding $\mathbb{R}P^3 \subset \mathbb{R}P^\infty$, which generates the third homology group and is dual to α. Hence, we have

$$\begin{aligned} Z(\mathbb{R}P^3) &= \frac{1}{2}\left(\langle\gamma_0(\alpha), [\mathbb{R}P^3]\rangle + \langle\gamma_1(\alpha), [\mathbb{R}P^3]\rangle\right) \\ &= \frac{1}{2}\left(1 + (-1)\right) \\ &= 0. \end{aligned}$$

To close, we now turn to the untwisted quantum invariants of closed 2-manifolds. In the untwisted case, the $\mathbb{Z}_2$ Chern-Simons quantum space associated to a 2-manifold Σ is

$$V_\Sigma = \frac{1}{|\mathbb{Z}_2|} \cdot L^2\Big(\mathrm{Hom}\big(\pi_1(M, x_i), \mathbb{Z}_2\big), \mathbb{Z}_2\Big)^{\mathbb{Z}_2}.$$

This is an immediate consequence of equation (6) and theorem 2.1 (see [4]). Here, the L^2 metric is defined via $\mu(P) = 1/|\mathrm{Aut}(P)|$, the prefactor $1/|\mathbb{Z}_2|$ multiplies this L^2 metric, and by $(\cdot)^{\mathbb{Z}_2}$ we mean the invariants under the $\mathbb{Z}_2$ action by conjugation.

As an example, let us calculate the quantum Hilbert space for S^2. In this case, we have that $\pi_1(M) = \pi_1\big(S^2\big) \cong \{1\}$. Thus, $\mathrm{Hom}\big(\pi_1(S^2), \mathbb{Z}_2\big) \cong \mathrm{Hom}\big(\{1\}, \mathbb{Z}_2\big)$. Hence, denoting the distinct homomorphism from $\{1\}$ to $\mathbb{Z}_2$ by $\star$ we have

$$V_{S^2} \cong \frac{1}{|\mathbb{Z}_2|} \cdot L^2\big(\star, \mathbb{Z}_2\big)^{\mathbb{Z}_2}.$$

Furthermore, there only exists one homomorphism from $\star$ to $\mathbb{Z}_2$, the one which sends $\star$ to the identity element in $\mathbb{Z}_2$. Note, this is incidentally invariant under conjugation by $\mathbb{Z}_2$. So, $V_{S^2} = 1/|\mathbb{Z}_2| \cdot L^2\big(\text{one pt. space}\big)$. Using the fact that $L^2\big(\text{one pt. space}\big)$ is a 1-dimensional Hilbert space and the fact that a 1-dimensional Hilbert space is canonically isomorphic to $\mathbb{C}$ (with the usual metric on $\mathbb{C}$), we arrive at the conclusion

$$V_{S^2} \cong \frac{1}{2} \cdot \mathbb{C}.$$

Here, $1/2$ multiplies the usual metric on $\mathbb{C}$.

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