Affine Lie Algebras

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January 16, 2008

Abstract

In these lectures the untwisted affine Lie algebras will be constructed. The reader is assumed to be familiar with the theory of semisimple Lie algebras, e.g. that he or she knows a big part of James E. Humphreys' *Introduction to Lie algebras and representation theory* [1]. The notations used in these notes will be taken from [1]. These lecture notes are based on the course Affine Lie Algebras given by Prof. Dr. Johan van de Leur at the Mathematical Research Insitute in Utrecht (The Netherlands) during the fall of 2007.

1 Semisimple Lie Algebras

1.1 Root Spaces

Recall some basic notions from [1]. Let L be a semisimple Lie algebra, H a Cartan subalgebra (CSA), and $\kappa(x,y) = Tr(\mathrm{ad}\,(x)\,\mathrm{ad}\,(y))$ the Killing form on L. Then the Killing form is symmetric, non-degenerate (since L is semisimple and using theorem 5.1 page 22 [1]), and associative;[‡] i.e. $\kappa : L \times L \to \mathbb{F}$ is bilinear on L and satisfies

$$\kappa\left([x,y],z\right) = \kappa\left(x,[y,z]\right) \, .$$

The restriction of the Killing form to the CSA, denoted by $\kappa_{|H}(\cdot,\cdot)$, is non-degenerate (Corollary 8.2 page 37 [1]). This allows for the identification of H with H^* (see [1] §8.2: to $\phi \in H^*$ there corresponds a unique element $t_\phi \in H$ satisfying $\phi(h) = \kappa(t_\phi, h)$ for all $h \in H$). This makes it possible to define a symmetric, non-degenerate bilinear form, $(\cdot,\cdot) : H^* \times H^* \to \mathbb{F}$, given on H^* as

$$(\alpha,\beta) = \kappa(t_\alpha, t_\beta) \qquad (\forall\ \alpha,\ \beta \in H^*\,)\,.$$

Let $\Phi \subset H^*$ be the root system corresponding to L and $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ a fixed basis of Φ (Δ is also called a *simple root system*). Then the set of roots Φ can be written as the

[‡]Instead of associative one sometimes uses the term *invariant*.

disjoint union $\Phi = \Phi^+ \cup \Phi^-$ of positive roots Φ^+, where $\beta \in \Phi^+$ implies that the expansion coefficients of β in the Δ basis are all non-negative integers (i.e. $\beta = \sum k_i \alpha_i$ with $k_i \geq 0$ and $\alpha_i \in \Delta$) and the set of negative roots $\Phi^- = -\Phi^+$. The *root space decomposition* of L is given by

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha = H \oplus \bigoplus_{\alpha \in \Phi^+} (L_\alpha \oplus L_{-\alpha})$$

where every root space L_α is 1-dimensional. The root space decomposition can be rearranged, see appendix A, to give the *triangular decomposition* of the Lie algebra L

$$L = N_- \oplus H \oplus N_+,$$

where

$$N_+ = N(\Delta) = \bigoplus_{\alpha \in \Phi^+} L_\alpha \quad \text{and} \quad N_- = \bigoplus_{\alpha \in \Phi^+} L_{-\alpha}.$$

It should be noted that the name triangle decomposition arises from the fact that N_- is represented by lower triangular matrices, N_+ is represented by upper triangular matrices, and H is represented by diagonal matrices.

Also one has that

$$[x, y] = \kappa(x, y) t_\alpha \tag{1.1}$$

for $x \in L_\alpha$ and $y \in L_{-\alpha}$.

1.2 The Weyl Group

The *Weyl group* $\mathcal{W}$ of L is the subgroup of $GL(H^*)$ generated by all reflections of the form

$$\sigma_\alpha(\lambda) = \lambda - \langle \lambda, \alpha \rangle \alpha \qquad (\forall\ \alpha \in \Phi,\ \lambda \in H^*)\,, \tag{1.2}$$

where

$$\langle \alpha, \beta \rangle = 2 \frac{(\alpha, \beta)}{(\beta, \beta)}.$$

One can show (see [1], §10.3) that $\mathcal{W}$ is in fact generated by all the *fundamental reflections*,

$$r_i = \sigma_{\alpha_i} \qquad (\forall\ \alpha_i \in \Delta)\,.$$

Also, one can form a matrix, known as the *Cartan matrix* of Φ, whose entries are given by $(\langle \alpha_i, \alpha_j \rangle)_{1 \leq i,j \leq \ell}$.‡

‡This definition of the Cartan matrix differs from the definition given in [2]. In particular, here Kac defines what is known as the *generalized Cartan matrix*. This will be defined later in the notes.

1.3 Serre's Theorem

Fix, for all $i = 1, \ldots, \ell$, a standard set of generators $x_i \in L_{\alpha_i}$ and $y_i \in L_{-\alpha_i}$, so that $[x_i, y_i] = h_i$. Then L is generated by the elements x_i, y_i and h_i, with $1 \leq i \leq \ell$, and they satsify the following relations:

$$\begin{aligned} [h_i, h_j] &= 0\,, \\ [x_i, y_j] &= \delta_{ij} h_i\,, \\ [h_i, x_j] &= \langle \alpha_j, \alpha_i \rangle x_j\,, \\ [h_i, y_j] &= -\langle \alpha_j, \alpha_i \rangle y_j\,, \end{aligned} \tag{1.3}$$

and the so-called *Serre relations*

$$\begin{aligned} (\mathrm{ad}\,(x_i))^{1-\langle \alpha_j, \alpha_i \rangle}(x_j) &= 0 \qquad (\text{for } i \neq j)\ , \\ (\mathrm{ad}\,(y_i))^{1-\langle \alpha_j, \alpha_i \rangle}(y_j) &= 0 \qquad (\text{for } i \neq j)\ . \end{aligned} \tag{1.4}$$

The converse of this statement also holds (see [1] §18.3), known as *Serre's theorem*.

Theorem 1.1 (Serre). *Let Φ, and Δ be as above and let L be the Lie algebra generated by the elements x_i, y_i and h_i, for $1 \leq i \leq \ell$, subject to the relations (1.3) and (1.4). Then L is a finite dimensional semisimple Lie algebra with CSA spanned by $\{h_i\}_{i=1}^{\ell}$ and corresponding root system Φ.*

1.4 Cartan Involution

Using this presentation, it is straightforward to define an involution† ω, called the *Cartan involution*, on L that interchanges the root spaces L_α and $L_{-\alpha}$. This mapping, $\omega : L \to L$, is defined by

$$\omega(x_i) = -y_i, \quad \omega(y_i) = -x_i, \quad \text{and} \quad \omega(h_i) = -h_i \tag{1.5}$$

A quick check shows that ω is indeed an involution. For example let $x \in L_\alpha$, then

$$\begin{aligned} (\omega \circ \omega)(x_i) &= \omega\left(\omega(x_i)\right) \\ &= \omega(-y_i) \\ &= -\omega(y_i) \\ &= -(-x_i) \\ &= x_i\,. \end{aligned}$$

†An involution is a mapping such that composition with itself is the identity mapping, i.e. if $f : X \to X$ is an involution, then $(f \circ f)(x) = x$ for all $x \in X$.

1.5 Concrete Example of the Above Notions

The previously developed theory will now be applied to the simple Lie algebra of type A_ℓ.

Example 1.1 $\mathfrak{sl}_n(\mathbb{C})$*:*

$\mathfrak{sl}_n(\mathbb{C})$ is the Lie algebra of all complex $n \times n$-matrices that have trace zero. Denote by e_{ij} the matrix which has a 1 on the i-th row and j-th column and 0 elsewhere. As CSA take the set of diagonal traceless matrices and define the killing form as $\kappa(x,y) = 2nTr(xy)$. The set of roots Φ consists of all $\epsilon_i - \epsilon_j$ with $1 \leq i,j \leq n$ and $i \neq j$, where $\epsilon_i(e_{jj}) = \delta_{ij}$. Now choose

$$\Delta = \{\alpha_i = \epsilon_i - \epsilon_{i+1} \,|\, 1 \leq i \leq n-1\}.$$

The element $t_{\alpha_i} \in H$ is equal to $\frac{1}{2n}(e_{ii} - e_{i+1,i+1})$, thus

$$(\alpha_i, \alpha_j) = \begin{cases} \frac{1}{n} & \text{if } i = j, \\ -\frac{1}{2n} & \text{if } |i-j| = 1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i-j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the $(n-1) \times (n-1)$ Cartan matrix is given by

$$C = \begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ 0 & & & -1 & 2 \end{pmatrix}$$

The elements h_i are given by $e_{ii} - e_{i+1,i+1}$ and the Weyl group is the permutation group of the elements ϵ_i, with r_i the reflection that interchanges ϵ_i and ϵ_{i+1}.

Choose $x_i = e_{i,i+1}$ and $y_i = e_{i+1,i}$, then it is straightforward to check that the relations (1.3) and (1.4) hold. As an example, for $j = i+1$, the first relation of 1.4, since $\langle \alpha_{i+1}, \alpha_i \rangle = -1$, gives

$$\begin{aligned} (ad(x_i))^2 (x_j) =& [x_i, [x_i, x_{i+1}]] \\ =& [e_{i,i+1}, [e_{i,i+1}, e_{i,i+2}]] \\ =& [e_{i,i+1}, e_{i,i+2}] \\ =& 0\,. \end{aligned}$$

The root spaces are $L_{\epsilon_i-\epsilon_j} = \mathbb{C}e_{ij}$ while N_+ consists of all strictly upper diagonal matrices and N_- the strictly lower diagonal matrices. The Cartan involution is given by $\omega(X) = -X^T$, where X^T stands for the transpose of the matrix X.

2 Central Extensions of a Lie Algebra

Let L be a Lie algebra over the complex field. The so called 1-*dimensional central extensions*, denoted by $\tilde{L}$, of the Lie algebra L over $\mathbb{C}$ are constructed as follows. First extend L, while viewing it as a vector space, by one dimension

$$\tilde{L} = L \oplus \mathbb{C}K\,.$$

Then define a new bracket on $\tilde{L}$, which will be denoted by $[\cdot,\cdot]_\circ$, in such a way that K is a central element, i.e.

$$[K,x]_\circ = 0, \qquad \text{for all} \quad x \in L\,.$$

So, for $x, y \in L$ and $\lambda, \mu \in \mathbb{C}$ define $[\cdot,\cdot]_\circ$, the Lie bracket turning $\tilde{L}$ into a Lie algebra, as

$$[x+\lambda K, y+\mu K]_\circ := [x,y] + \psi(x,y)K, \tag{2.1}$$

where the brackets $[\cdot,\cdot]$ appearing on the right hand side are defined on L not $\tilde{L}$ and $\psi : L \times L \to \mathbb{C}$. Since $\tilde{L}$ has to be a Lie algebra, one has that the new bracket $[\cdot,\cdot]_\circ$ is bilinear, anti-symmetric and must satisfy the Jacobi identity. This leads to the condition that ψ must be a $\mathbb{C}$-valued bilinear function that satisfies

$$\psi(y,x) = -\psi(x,y)$$

and

$$\psi([x,y],z) + \psi([y,z],x) + \psi([z,x],y) = 0\,, \qquad \text{for all} \quad x,\ y, \text{ and } z \in L\,.$$

Such a $\mathbb{C}$-valued bilinear function ψ is called a 2-*cocycle* on L. A 2-cocycle is called *trivial*, or is called a 2-coboundary, if there exists a linear function $f : L \to \mathbb{C}$ such that

$$\psi(x,y) = f([x,y]) \qquad \text{for all} \quad x,y \in L\,.$$

Exercise 2.1 *Consider the case that $L = \mathfrak{sl}_2(\mathbb{C})$. Show that any cocycle ψ on $\mathfrak{sl}_2(\mathbb{C})$ is trivial. This can actually be extended, in this way, to any simple finite dimensional Lie algebra, i.e. any 2-cocycle is trivial on simple finite dimensional Lie algebras.*

Exercise 2.2 *Let L be a Lie algebra which possesses a symmetric invariant $\mathbb{C}$-valued* *bilinear form* $(\cdot,\cdot)$. *Let* d *be a derivation of* L *that satisfies* $(d(a),b) = -(a,d(b))$. *Show that* $\psi(a,b) = (d(a),b)$ *defines a cocycle on* L.

Exercise 2.3 *Let* L *be the Witt algebra, see appendix B, with basis* L_n (*for* $n \in \mathbb{Z}$) *and commutation relations given by*

$$[L_m, L_n] = (m-n)L_{n+m}\,.$$

Show that there is, up to trivial cocycles, only one non-trivial central extension of the Witt algebra and that it is given by

$$[L_m, L_n] = (m-n)L_{n+m} + \frac{K}{12}(m(m^2-1)\delta_{m,-n}\,, \tag{2.2}$$

where K *is the central element. Hint: If* $\psi(\cdot,\cdot)$ *is the cocycle, then denote by* $c_{m,n}$ *the element* $\psi(L_m, L_n)$. *Choose a new basis* $L'_n = L_n + \frac{1}{n}c_{n,0}K$, *for* $n \neq 0$ *and* $L'_0 = L_0 + \frac{1}{2}c_{1,-1}K$.

NOTE: The central extension of the Witt algebra is called the Virasoro algebra and it plays an important role in Conformal Field Theory.

3 The Loop Algebra

Let L be a semisimple Lie algebra. A *loop* is a map from the circle S^1 (paramterized by $e^{i\theta}$) to the Lie algebra L. It will be assumed that the loop is a polynomial. Thus making it possible to expand it in terms of a finite Fourier series,

$$g(\theta) = \sum_{n=-N}^{N} g_n e^{in\theta}, \quad \text{with} \quad g_n \in L\,.$$

Denote by $\bar{L}$ the vector space of all such maps. Then $\bar{L}$ becomes a Lie algebra, called the *loop algebra of* L, with Lie bracket $[\cdot,\cdot]_{\bar{L}}$ defined by

$$[e^{im\theta}g, e^{in\theta}h]_{\bar{L}} := e^{i(m+n)\theta}[g,h]$$

where the bracket on the right is the Lie bracket defined on L. For simplicity write, from now on, t instead of $e^{i\theta}$ and $g(t)$ instead of $g(\theta)$. With this definition $\bar{L}$, as a vector space, is given by

$$\bar{L} = \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}} L\,,$$

where $\mathbb{C}[t, t^{-1}]$ is the algebra of Laurent polynomials in t, and $[\cdot,\cdot]_{\bar{L}}$ by

$$[t^n \otimes g, t^m \otimes h]_{\bar{L}} = t^{m+n} \otimes [g, h].$$

4 Central Extension of the Loop Algebra

The 1-dimensional central extension of the loop algebra $\bar{L}$ will now be comstructed, where as before $\bar{L} = \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} L$.

Let $\kappa(\cdot,\cdot)$ be the Killing form on L[‡]. It can then be extended to a non-degenerate symmetric, invariant bilinear form $\kappa_{\bar{L}}(\cdot,\cdot)$ on $\bar{L}$ by

$$\kappa_{\bar{L}}(P(t) \otimes x, Q(t) \otimes y) = (P(t)Q(t)|_{t=0})\, \kappa(x, y)\,. \tag{4.1}$$

Note that

$$\kappa_{\bar{L}}(P(t) \otimes x, Q(t) \otimes y) = \left(\frac{1}{2\pi} \int_0^{2\pi} P(e^{i\theta}) Q(e^{i\theta})\, d\theta \right) \kappa(x, y).$$

Let $d = t\frac{d}{dt}$ be the derivation of $\mathbb{C}[t, t^{-1}]$, then one has the following result.

Lemma 4.1 *On $\bar{L}$ one has a $\mathbb{C}$-valued 2-cocycle that is defined by*

$$\psi_{\bar{L}}(P(t) \otimes x, Q(t) \otimes y) = \kappa_{\bar{L}}(d(P(t)) \otimes x, Q(t) \otimes y)_0\,.$$

Proof. See Exercise 2.2. For an example of calculating the 2-cocycle see appendix C. □

The 2-cocycle defined in Lemma 4.1 is non-trivial (Exercise 4.2), thus giving a non-trivial central extension $\tilde{L}$ of $\bar{L}$. As a vector space $\tilde{L}$ is given by

$$\tilde{L} = \bar{L} \oplus \mathbb{C}K = \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} L \oplus \mathbb{C}K$$

and this becomes a Lie algebra via

$$[P(t) \otimes x + \lambda K, Q(t) \otimes y + \mu K]_{\tilde{L}} = [P(t) \otimes x, Q(t) \otimes y]_{\bar{L}} + \psi_{\bar{L}}(P(t) \otimes x, Q(t) \otimes y) K\,.$$

It will be convenient to extend this algebra with one extra dimension. For this first extend the derivation d on $\mathbb{C}[t, t^{-1}]$ to a derivation on $\tilde{L}$ by

$$d_{\tilde{L}}(P(t) \otimes x + \lambda K) = d(P(t)) \otimes x\,,$$

and introduce

$$\hat{L} = \tilde{L} \oplus \mathbb{C}d = \bar{L} \oplus \mathbb{C}K \oplus \mathbb{C}d\,, \tag{4.2}$$

[‡]It is known (see [1] page 22) that $\kappa(\cdot,\cdot)$ is non-degenerate since the Lie algebra L is semisimple.

$\hat{L}$ becomes a Lie algebra when multiplication, $[\cdot,\cdot]_{\hat{L}}$, is defined as

$$[P(t)\otimes x+\lambda K+\mu d, Q(t)\otimes y+\nu K+\sigma d]_{\hat{L}} := P(t)Q(t)\otimes[x,y]+\mu d(Q(t))\otimes y-\sigma d(P(t))\otimes x+\psi_{\bar{L}}(P(t)\otimes x, Q(t)\otimes y)K\,.$$

The bilinear form $\kappa_{\bar{L}}(\cdot,\cdot)$, given by (4.1), on $\bar{L}$ can be extended to a non-degenerate symmetric bilinear form $\kappa_{\hat{L}}(\cdot,\cdot)$ on $\hat{L}$ by defining

$$\begin{aligned}
\kappa_{\bar{L}}(P(t)\otimes x, Q(t)\otimes y) &= \kappa_{\bar{L}}(P(t)\otimes x, Q(t)\otimes y)_0\,,\\
\kappa_{\bar{L}}(P(t)\otimes x, K) &= \kappa_{\bar{L}}(P(t)\otimes x, d) = 0\,,\\
\kappa_{\bar{L}}(K,K) &= \kappa_{\bar{L}}(d,d) = 0\,,\\
\kappa_{\bar{L}}(K,d) &= \kappa_{\bar{L}}(d,K) = 1\,.
\end{aligned}$$

Let H be the CSA of L. The restriction of $\kappa_{\hat{L}}(\cdot,\cdot)$ to $H\oplus\mathbb{C}K\oplus\mathbb{C}d$ is non-degenerate. In appendix D the Lie algebra $\hat{L}$ is constructed for the case when L is given by $\mathfrak{sl}_2(\mathbb{C})$.

Exercise 4.1 *Show that the 2-cocycle defined in Lemma 4.1 satisfies the conditions of exercise 2.2.*

Exercise 4.2 *Show that the 2-cocycle defined in Lemma 4.1 is non-trivial. .*

5 Untwisted Affine Lie Algebras

Now it will be shown for the Lie algebra $\hat{L}$ that one can find a set of generators x_i, y_i and h_i such that the relations (1.3) and (1.4) hold. This will be shown for the case that L is simple. The construction for L semisimple will then be immediately obvious.

Assume from now on in this section that L is a simple Lie algebra. Choose as in section 1 a simple root system Δ and let x_i,y_i and h_i, $1\le i\le \ell$, be the standard set of generators for L that satisfy (1.3) and (1.4). Let θ be the highest root of L. Then since L_θ is one dimensional one can choose an element $y'\in L_\theta$ such that $\kappa(y',\omega(y')) = -\frac{2}{(\theta,\theta)}$ and define $x' = -\omega(y')$. Now (see (1.1)), choose

$$h' = [x',y'] = -\kappa(x',y')t_\theta = -\frac{2t_\theta}{(\theta,\theta)}\,,$$

then

$$\begin{aligned}[h', x_i] &= -\frac{2}{(\theta,\theta)}[t_\theta, x_i] \\ &= -\frac{2}{(\theta,\theta)}\alpha_i(t_\theta)x_i \\ &= -\frac{2}{(\theta,\theta)}\kappa(t_{\alpha_i}, t_\theta)x_i \\ &= -\frac{2(\alpha_i,\theta)}{(\theta,\theta)}x_i \\ &= -\langle\alpha_i,\theta\rangle x_i\,. \end{aligned} \tag{5.1}$$

and

$$\begin{aligned}[h_i, x'] &= -\theta(h_i)x' \\ &= -\langle\theta,\alpha_i\rangle x' \end{aligned} \tag{5.2}$$

Let $\hat{L}$ be as in (4.2). Define the affine CSA

$$\hat{H} = H \oplus \mathbb{C}K \oplus \mathbb{C}d$$

and let $\delta \in \hat{H}^*$ be such that

$$\delta(h_i) = 0, \quad \delta(K) = 0 \quad \text{and } \delta(d) = 1\,.$$

Choose

$$x_i = t^0 \otimes x_i, \quad y_i = t^0 \otimes y_i \quad h_i = t^0 \otimes h_i, \qquad \text{for} \quad 1 \le i \le \ell\,, \tag{5.3}$$

and

$$x_0 = t \otimes x', \quad y_0 = t^{-1} \otimes y', \quad h_0 = t^0 \otimes h' + \frac{2}{(\theta,\theta)}K\,. \tag{5.4}$$

Now, define $\alpha_0 \in \hat{H}^*$ by

$$\alpha_0 = \delta - \theta\,. \tag{5.5}$$

then one can show that these x_i, y_i and h_i for $i = 0, 1, \ldots, \ell$ satisfy the equations (1.3) and (1.4) and that they generate $\tilde{L}$.

The (extended) root system of $\hat{L}$ is equal to

$$\hat{\Phi} = \{j\delta + \alpha \,|\, j \in \mathbb{Z},\ \alpha \in \Phi\} \cup \{j\delta \,|\, j \in \mathbb{Z},\ j \neq 0\}\,. \tag{5.6}$$

Note that we no longer have that any root has nonzero length. In this case there are so called *imaginary roots*, i.e., roots $\beta \in \hat{\Phi}$ for which $(\beta, \beta) \leq 0$. The roots $j\delta$ have

$$(j\delta, j\delta) = 0\,.$$

All other roots β are so called *real roots*, they have $(\beta, \beta) > 0$. In particular

$$(j\delta + \alpha, j\delta + \alpha) = (\alpha, \alpha) > 0\,.$$

So we have the following decomposition in real and imaginary roots:

$$\hat{\Phi} = \hat{\Phi}^{re} \cup \hat{\Phi}^{im} \quad \text{(disjoint union)},$$

where

$$\hat{\Phi}^{re} = \{j\delta + \alpha \,|\, j \in \mathbb{Z},\ \alpha \in \Phi\} \quad \text{and} \quad \hat{\Phi}^{im} = \{j\delta \,|\, j \in \mathbb{Z},\ j \neq 0\}\,.$$

The simple root system of $\hat{L}$ is

$$\hat{\Delta} = \{\alpha_0, \alpha_1, \ldots, \alpha_\ell\}\,,$$

hence

$$\hat{\Phi}^+ = \{\alpha \,|\, \alpha \in \Phi^+\} \cup \{j\delta + \alpha \,|\, j > 0,\ \alpha \in \Phi\} \cup \{j\delta \,|\, j > 0\}, \quad \hat{\Phi}^- = -\hat{\Phi}^+$$

and $\hat{\Phi} = \hat{\Phi}^+ \cup \hat{\Phi}^-$ (disjoint union). As for finite dimensional simple Lie algebras we have a root space decomposition

$$\hat{L} = \hat{H} \oplus \bigoplus_{\alpha \in \hat{\Phi}} \hat{L}_\alpha = \hat{H} \oplus \bigoplus_{\alpha \in \hat{\Phi}^+} \left(\hat{L}_\alpha \oplus \hat{L}_{-\alpha}\right)$$

and a triangular decomposition

$$\hat{L} = \hat{N}_- \oplus \hat{H} \oplus \hat{N}_+, \qquad \text{where}$$
$$\hat{N}_+ = \hat{N}(\hat{\Delta}) = \bigoplus_{\alpha \in \hat{\Phi}^+} \hat{L}_\alpha \quad \text{and } N_- \bigoplus_{\alpha \in \hat{\Phi}^+} \hat{L}_{-\alpha}\,.$$

The matrix $(\langle \alpha_i, \alpha_j \rangle)_{0 \leq i,j \leq \ell}$ is called the the *extended Cartan matrix* of L or the *Cartan matrix* of $\hat{L}$. In figure we give the extended Dynkin diagrams corresponding to the extended Cartan matrix of L. The x encodes the root α_0 and the black nodes correspond to the short roots and the white nodes to the long roots.

In analogy with the semisimple case one has (see e.g. [2]):

Theorem 5.1 (Kac-Gabber). *Let $\hat{\Phi}$, and $\hat{\Delta}$ be as above. Let $\hat{L}$ be the Lie algebra generated by the elements x_i, y_i, h_i, for $0 \leq i \leq \ell$ and d, subject to the relations (1.3) and*

(1.4) and

$$[d,h_i]=[d,x_i]=[d,y_i]=0 \qquad \textit{for} \quad i\neq 0,$$
$$[d,h_0]=0, \quad [d,x_0]=x_0, \quad \textit{and} \quad [d,y_0]=-y_0\,,$$

then $\hat{L}$ *is an untwisted affine Lie algebra, with CSA spanned by the* h_i *and* d *and with corresponding root system* $\hat{\Phi}$*.*

We can extend the Cartan involution ω on L to a Cartan involution $\hat{\omega}$ on $\hat{L}$, by putting for $i=1,2,\dots,\ell$

$$\hat{\omega}(x_i)=\omega(x_i)=-y_i, \quad \hat{\omega}(y_i)=\omega(y_i)=-x_i, \quad \hat{\omega}(h_i)=\omega(h_i)=-h_i$$

and

$$\hat{\omega}(x_0)=\hat{\omega}(tx')=-t^{-1}y'=-y_0, \quad \hat{\omega}(y_0)=\hat{\omega}(t^{-1}y')=-tx'=-x_0, \quad \hat{\omega}(d)=-d.$$

Clearly, since $\hat{\omega}(h')=-h'$ we find that also $\hat{\omega}(K)=-K$.

The affine Weyl group $\hat{W}$ of $\hat{L}$ is the subgroup of $GL(\hat{H}^*)$ generated by all reflections ($\lambda\in\hat{H}^*$)

$$\sigma_\beta(\lambda)=\lambda-\langle\lambda,\beta\rangle\beta, \qquad \text{for all } \beta\in\hat{\Phi}^{re}\,. \tag{5.7}$$

One can show that this group is generated by all simple reflections $r_i=\sigma_{\alpha_i}$, for $i=0,1,\dots.\ell$. Note that $w(\delta)=\delta$ for any $w\in\hat{W}$.

Let for $\alpha\in\Phi$

$$\alpha^\vee=2\frac{\alpha}{(\alpha,\alpha)}$$

and introduce for α a long root the linear map $t_{\alpha^\vee}$ by

$$t_{\alpha^\vee}(\lambda)=\lambda+(\lambda,\delta)\alpha^\vee-\left((\lambda,\alpha^\vee)+\frac{1}{2}(\alpha^\vee,\alpha^\vee)(\lambda,\delta)\right)\delta\,,$$

for $\lambda \in H^*$. Then

$$\begin{aligned}\sigma_{\alpha_0}\sigma_\theta(\lambda) =&\sigma_{\delta-\theta}\left(\lambda - \langle\lambda,\theta\rangle\theta\right)\\ =&\sigma_{\delta-\theta}\left(\lambda\right) - \langle\lambda,\theta\rangle\sigma_{\delta-\theta}\left(\theta\right)\\ =&\lambda - \langle\lambda,\delta-\theta\rangle(\delta-\theta) - \langle\lambda,\theta\rangle\theta - 2\langle\lambda,\theta\rangle(\delta-\theta)\\ =&\lambda - \frac{2(\lambda,\delta)}{(\theta,\theta)}(\delta-\theta) - \langle\lambda,\theta\rangle(\delta-\theta) - \langle\lambda,\theta\rangle\theta\\ =&\lambda - \frac{2(\lambda,\delta)}{(\theta,\theta)}(\delta-\theta) - \langle\lambda,\theta\rangle\delta\\ =&\lambda + (\lambda,\delta)\theta^\vee - \left((\lambda,\theta^\vee) + \frac{1}{2}(\theta^\vee,\theta^\vee)(\lambda,\delta)\right)\delta\\ =&t_{\theta^\vee}(\lambda)\,.\end{aligned}$$

Let $w \in W$, then

$$\begin{aligned}wt_{\alpha^\vee}w^{-1}(\lambda) =&w\left(w^{-1}(\lambda) + (w^{-1}(\lambda),\delta)\alpha^\vee - \left((w^{-1}(\lambda),\alpha^\vee) + \frac{1}{2}(\alpha^\vee,\alpha^\vee)(w^{-1}(\lambda),\delta)\right)\delta\right)\\ =&\lambda + (\lambda,\delta)w(\alpha^\vee) - \left((\lambda,w(\alpha^\vee)) + \frac{1}{2}(w(\alpha^\vee),w(\alpha^\vee))(\lambda,\delta)\right)\delta\\ =&t_{w(\alpha^\vee)}(\lambda)\,,\end{aligned} \tag{5.8}$$

since $w(\delta) = \delta$ and $\langle w(\alpha), w(\beta)\rangle = \langle\alpha,\beta\rangle$. Since the highest root θ of a simple Lie algebra L is always a long root and the Weyl group acts transitively on all the long roots, we obtain all $t_{\alpha^\vee}$ with α a long root from $t_{\theta^\vee}$ by conjugation with all $w \in W$. Note also that

$$\begin{aligned}t_{\alpha^\vee}t_{\beta^\vee}(\lambda) =&t_{\alpha^\vee}(\left(\lambda + (\lambda,\delta)\beta^\vee - \left((\lambda,\beta^\vee) + \frac{1}{2}(\beta^\vee,\beta^\vee)(\lambda,\delta)\right)\delta\right)\\ =&t_{\alpha^\vee}(\lambda) + (\lambda,\delta)t_{\alpha^\vee}(\beta^\vee) - \left((\lambda,\beta^\vee) + \frac{1}{2}(\beta^\vee,\beta^\vee)(\lambda,\delta)\right)\delta\\ =&\lambda + (\lambda,\delta)\alpha^\vee - \left((\lambda,\alpha^\vee) + \frac{1}{2}(\alpha^\vee,\alpha^\vee)(\lambda,\delta)\right)\delta\\ &\quad + (\lambda,\delta)\left(\beta^\vee + (\beta^\vee,\delta)\alpha^\vee - \left((\beta^\vee,\alpha^\vee)\frac{1}{2}(\alpha^\vee,\alpha^\vee)(\beta^\vee,\delta)\right)\delta\right)\\ &\quad - \left((\lambda,\beta^\vee) + \frac{1}{2}(\beta^\vee,\beta^\vee)(\lambda,\delta)\right)\delta\\ =&\lambda + (\lambda,\delta)(\alpha^\vee+\beta^\vee) - \left((\lambda,\alpha^\vee+\beta^\vee) + \frac{1}{2}(\alpha^\vee+\beta^\vee,\alpha^\vee+\beta^\vee)(\lambda,\delta)\right)\delta\\ =&t_{\alpha^\vee+\beta^\vee}(\lambda)\,.\end{aligned}$$

Let T be the subgroup of $GL(\hat{H}^*)$ generated by all $t_{\alpha^\vee}$ for α a long root of L, then an element of T can be written as $t_{\beta^\vee}$ with β in the lattice generated by all long roots. T is called the *group of translations* and

Proposition 5.1

$$\hat{W} = W \ltimes T\,.$$

Proof. From the above considerations it is clear that $t_{\alpha^\vee} \in T$ is also an element of $\hat{W}$. Since it has infinite order it cannot be an element of W, except when $\alpha = 0$. Thus $W \cup T = \{1\}$, and since $w t_{\alpha^\vee} w^{-1} = t_{w(\alpha^\vee)}$, we find that T is a normal subgroup of $\hat{W}$. Since $\hat{W}$ is generated by all simple reflections r_i for $i = 0, 1, \dots \ell$, and $r_i \in W$ for $i \neq 0$ and $r_0 = t_{\theta^\vee}\sigma_\theta$, we find also that $\hat{W} \subset W \ltimes T$. □

Exercise 5.1 *Use the definition (5.5) of α_0 to calculate the extended Cartan matrix for the cases that L is a simple Lie algebra of type A, B, C and D.*

Exercise 5.2 *Show that the root system of $\hat{L}$ is equal to (5.6) and give the dimension of the corresponding root spaces.*

Exercise 5.3 *Express the elements $j\delta + \epsilon_k - \epsilon_\ell$ and $j\delta$ in $\hat{\Phi}$ of the affine Lie algebra $\hat{\mathfrak{sl}}(n, \mathbb{C})$ in terms of $\alpha_0,\ \alpha_1, \dots,\ \alpha_{n-1}$.*

Exercise 5.4 *Show that the elements x_i, y_i and h_i for $i = 0, 1, \dots, \ell$, defined in (5.3) and (5.4) generate the Lie algebra $\tilde{L}$.*

Exercise 5.5 *Show that the elements x_i, y_i and h_i for $i = 0, 1, \dots, \ell$, defined in (5.3) and (5.4) satisfy the equations (1.3) and (1.4).*

6 Kac-Moody algebra's

We will now use the abstract formulation of the semisimple and untwisted affine Lie algebras as given in Theorem 1.1 and Theorem 5.1 and generalize this, thus obtaining certain infinite dimensional Lie algebras. As a reference we refer to the book of Victor Kac [2]. Note however that our notations are somewhat different and our definition of a Kac-Moody Lie algebra is also different. When the generalized Cartan matrix is symmetrizable both definitions lead to the same Lie algebra.

We start with the definition of a generalized Cartan matrix. We call a complex $n \times n$-matrix $C = (c_{ij})_{1 \le i,j \le n}$ of rank ℓ a *generalized Cartan matrix* if the following conditions hold:

$$\begin{aligned} c_{ii} = 2 \qquad &\text{for all} \quad 1 \le i \le n, \\ 0 \ge c_{ij} \in \mathbb{Z} \qquad &\text{for} \quad i \neq j, \\ c_{ij} = 0 \qquad &\text{implies} \quad c_{ji} = 0. \end{aligned} \tag{6.1}$$

Such a generalized Cartan matrix is called *symmetrizable* if there exists an invertible diagonal matrix $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ and a symmetric matrix $B = (b_{ij})_{1\le i,j\le n}$ such that

$$C = BD\,. \tag{6.2}$$

Of course such a matrix D and B are not unique, however we will always assume that all $0 < d_i \in \mathbb{Q}$. *From now on we will always assume that a generalized Cartan matrix is symmetrizable.*

In both semisimple and affine case one can show (see exercise 6.1) that the (extended) Cartan matrix is symmetrizable. In that case B more or less defines a bilinear form on the linear span of the simple roots, which is a subspace of H^*.

Now let H be a complex $2n-\ell$-dimensional vector space and let

$$\Delta^\vee = \{h_1, h_2, \ldots, h_n\} \subset H\,,$$

be a linearly independent subset. Define

$$H' = \sum_{i=1}^{n} \mathbb{C}h_i\,.$$

Since we can identify H with H^*, we define the subset

$$\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset H^*$$

and a non-degenerate pairing $\langle\cdot,\cdot\rangle : H^* \times H \to \mathbb{C}$ such that

$$\langle \alpha_i, h_j\rangle = c_{ij}\,. \tag{6.3}$$

Note that since the rank of C is ℓ and the dimension of H' is n, we need the extra $n-\ell$-dimensions in H to make the α_i linearly independent. We decompose H as follows

$$H = H' \oplus H''$$

where H'' is an $n-\ell$-dimensional complementary subspace to H'. Define a symmetric non-degenerate bilinear form $(\cdot,\cdot)$ on H by

$$(h_i, h) = d_i\langle \alpha_i, h\rangle \qquad \text{and} \quad (H', H'') = 0\,,$$

then

$$(h_i, h_j) = d_i c_{ij} = d_i d_j b_{ij} \tag{6.4}$$

Using this nondegenerate bilinear form on H we define an isomorphism $\nu : H \to H^*$ by

$$\langle \nu(h), k \rangle = (h, k) \qquad \text{for all} \quad k \in H\,.$$

This makes it possible to define a symmetric, non-degenerate, bilinear form on H^*, by

$$(\nu(h), \nu(k)) = (h, k)\,.$$

Then from (6.3) and (6.4) we deduce that

$$\nu(h_i) = d_i \alpha_i$$

and thus that

$$(\alpha_i, \alpha_j) = b_{ij}\,.$$

Note that if we follow the definition of Humphreys [1] and define $\langle \alpha_i, \alpha_j \rangle = 2\frac{(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$ then

$$\langle \alpha_i, \alpha_j \rangle = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} = 2\frac{b_{ij}}{b_{jj}} = 2\frac{c_{ij}}{c_{jj}} = c_{ij} \tag{6.5}$$

We can now introduce the *Kac-Moody lie algebra* $\mathfrak{g}(C)$ associated to the matrix C.

Definition 6.1 *Let C be a symmetrizable generalized Cartan matrix and H, Δ and $\Delta^\vee$ be as above. The Lie algebra $\mathfrak{g}(C)$ is the Lie algebra with generators x_i, y_i, $i = 1, 2, \ldots n$ and H and relations*

$$\begin{aligned} [h, k] &= 0 \qquad \textit{for all} \quad h, k \in H, \\ [x_i, y_j] &= \delta_{ij} h_i\,, \\ [h, x_j] &= \langle \alpha_j, h \rangle e_j\,, \\ [h_i, y_j] &= -\langle \alpha_j, h \rangle f_j\,. \end{aligned} \tag{6.6}$$

and the Serre relations

$$\begin{aligned} (\operatorname{ad} x_i)^{1-c_{ji}}(x_j) &= 0\,, \\ (\operatorname{ad} y_i)^{1-c_{ji}}(y_j) &= 0\,. \end{aligned} \tag{6.7}$$

Note that this definition is not the definition that is given by Kac in [2]. First of all Kac has a matrix A which is the transposed of C. Also the Serre relations (6.7) are replaced by some other condition. Kac and Gabber have shown (see e.g. [2]) that when C is symmetrizable then Definition 6.6 gives the Kac-Moody Lie algebra as it is defined in [2]. If C is nonsymmetrizable, then until now it is not known if both definitions define the same Lie algebra.

We call H the *Cartan subalgebra* (CSA) of $\mathfrak{g}(C)$. Note that

$$[h_i, x_j] = c_{ji} x_j \qquad \text{and} \quad [h_i, y_j] = -c_{ji} y_j \,,$$

If C is the Cartan matrix of a semisimple Lie algebra, than the rank of C is n and thus the corresponding Lie algebra $\mathfrak{g}(C)$ gives exactly the construction of L as in Theorem 1.1.

Let $Q = \sum_{i=1}^n \mathbb{Z}\alpha_i$ be the *root lattice of* $\mathfrak{g}(C)$. One can show, in a similar way as is done in Serre's Theorem in the finite dimensional case, that $\mathfrak{g}(C)$ is the direct sum of H together with the linear span of the elements $[x_{i_1}, [x_{i_2}, [\cdots, [x_{i_{p-1}}, x_{i_p}] \ldots]$ and $[y_{i_1}, [y_{i_2}, [\cdots, [y_{i_{p-1}}, y_{i_p}] \ldots]$. Thus, if we define $Q_+ = \sum_{i=1}^n \mathbb{Z}_+ \alpha_i$, where $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, then we have the following *triangular decomposition*

$$\mathfrak{g}(C) = N_- \oplus H \oplus N_+, \qquad \text{where} \quad N_\pm = \bigoplus_{\alpha \in Q_+,\ \alpha \neq 0} \mathfrak{g}_\alpha \,.$$

Note that the space $\mathfrak{g}_\alpha$ is finite dimensional. This is easy to see, let $\alpha = \sum_i k_i \alpha_i$, then the *height* $ht(\alpha)$ of α is $ht(\alpha) = \sum_i k_i$. Since all α are either in Q_+ or $-Q_+$, all $k_i \geq 0$ or all $k_i \leq 0$. If all $k_i \geq 0$, then $\mathfrak{g}_\alpha$ is the linear span of all elements $[x_{i_1}, [x_{i_2}, [\cdots, [x_{i_{ht(\alpha-1)}}, x_{i_{ht(\alpha)}}] \ldots]$, with $\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_{ht(\alpha)}} = \alpha$. A similar condition holds when all $k_i \leq 0$. Thus

$$\dim(\mathfrak{g}_\alpha) \leq n^{|ht(\alpha)|} \,.$$

An element $\alpha \in Q$ is called a *root* if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq \{0\}$. The set of all roots is denoted by Φ and one can decompose this set in the disjoint union

$$\Phi = \Phi_+ \cup \Phi_- \,,$$

of the positive roots $\Phi_+ = \Phi \cap Q_+$ and the negative roots $\Phi_- = \Phi \cap -Q_+$. As in the semisimple case we have a Cartan involution ω, which is defined by:

$$\omega(x_i) = -y_i, \quad \omega(y_i) = -x_i \quad \text{and} \quad \omega(h) = -h \text{ for all } h \in H \,.$$

using this Cartan involution it is clear that $\Phi_- = -\Phi_+$ and that $\dim \mathfrak{g}_{-\alpha} = \dim \mathfrak{g}_\alpha$.

It is possible to extend the bilinear form $(\cdot, \cdot)$ on H to a non-degenerate symmetric invariant bilinear form on $\mathfrak{g}(C)$, by defining it on $\oplus_i (\mathbb{C}x_i \oplus \mathbb{C}y_i)$ by

$$(x_i, y_j) = \delta_{ij} d_i, \quad (x_i, x_j) = (y_i, y_j) = 0, \quad (x_i, h) = (y_i, h) = 0 \quad \text{for all } h \in H \,. \qquad (6.8)$$

We now check part of the invariance:

$$([x_i, y_j], h_k) = \delta_{ij}(h_j, h_k) = \delta_{ij} d_j d_k b_{jk} = \delta_{ij} d_i c_{jk} = (x_i, [y_j, h_k]) .$$

Using induction on the height of the roots (see also exercise 6.2), symmetry and invariance, one can extend this bilinear form to a non-degenerate form on the whole $\mathfrak{g}(C)$, such that

$$(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0 \qquad \text{for } \alpha + \beta \neq 0 . \tag{6.9}$$

We define the *Weyl group* of $\mathfrak{g}(C)$ as the subgroup of $GL(H^*)$ generated by all fundamental reflections r_i for $i = 1, 2, \ldots, n$:

$$r_i(\lambda) = \lambda - \langle \lambda, h_i \rangle \alpha_i, \qquad \text{for all} \quad \lambda \in H^* .$$

Using exercise 6.3, one can show that all x_i and y_i act locally nilpotent on $\mathfrak{g}(C)$, i.e., $(\text{ad}x_i)^k(z) = 0$ for $k >> 0$ and any $z \in \mathfrak{g}(C)$ (the same holds for y_i). We can lift the fundamental reflections to the whole Kac-Moody algebra. Define

$$\hat{r}_i = \exp(\text{ad}y_i) \exp(-\text{ad}x_i) \exp(\text{ad}y_i) ,$$

then

$$\hat{r}_i(\mathfrak{g}_\alpha) = \mathfrak{g}_{r_i(\alpha)} .$$

Exercise 6.1 *Show that Cartan matrix of both a semisimple and an untwisted affine Lie algebra is symmetrizable and show that one can choose* $d_i = \frac{2}{(\alpha_i, \alpha_i)}$.

Exercise 6.2 *Show that for the non-degenerate symmetric invariant bilinear form defined by (6.4) and (6.8) on* $\mathfrak{g}(C)$ *condition (6.9) holds whenever* $ht(\alpha) + ht(\beta) \neq 0$.

Exercise 6.3 *Let* $L = \mathfrak{g}(C)$ *be a Kac-Moody algebra. Show that if there exists* $m, n > 0$ *such that* $(\text{ad}x)^m(y) = 0$ *and* $(\text{ad}x)^n(z) = 0$ *for* $x, y, z \in L$*, that there exists* $k > 0$ *such that also* $(\text{ad}x)^k([y, z]) = 0$. *Using this, show that all generators* x_i *and* y_i*, for* $i = 1, 2, \ldots, n$ *act locally nilpotent on* $N_- \oplus N_+$

Appendix A

Triangular Decomposition of L

Consider the root space decomposition of the Lie algebra L

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha .$$

This can be written as (using the commutivity of the direct sum)

$$\begin{aligned}
L &= H \oplus \bigoplus_{\alpha\in\Phi^+} (L_\alpha \oplus L_{-\alpha}) \ , \\
&= H \oplus (L_{\alpha_1} \oplus L_{-\alpha_1} \oplus L_{\alpha_2} \oplus L_{-\alpha_2} \oplus \cdots \oplus L_{\alpha_n} \oplus L_{-\alpha_n}) \ , \\
&= H \oplus L_{\alpha_1} \oplus L_{-\alpha_1} \oplus L_{\alpha_2} \oplus L_{-\alpha_2} \oplus \cdots \oplus L_{\alpha_n} \oplus L_{-\alpha_n} \ , \\
&= H \oplus L_{-\alpha_1} \oplus L_{\alpha_1} \oplus L_{-\alpha_2} \oplus L_{\alpha_2} \oplus \cdots \oplus L_{-\alpha_n} \oplus L_{\alpha_n} \ , \\
&= L_{-\alpha_1} \oplus H \oplus L_{\alpha_1} \oplus L_{-\alpha_2} \oplus L_{\alpha_2} \oplus \cdots \oplus L_{-\alpha_n} \oplus L_{\alpha_n} \ , \\
&= L_{-\alpha_1} \oplus H \oplus L_{-\alpha_2} \oplus L_{\alpha_1} \oplus L_{\alpha_2} \oplus \cdots \oplus L_{-\alpha_n} \oplus L_{\alpha_n} \ , \\
&= L_{-\alpha_1} \oplus L_{-\alpha_2} \oplus H \oplus L_{\alpha_1} \oplus L_{\alpha_2} \oplus \cdots \oplus L_{-\alpha_n} \oplus L_{\alpha_n} \ , \\
&= L_{-\alpha_2} \oplus L_{-\alpha_1} \oplus H \oplus L_{\alpha_1} \oplus L_{\alpha_2} \oplus \cdots \oplus L_{-\alpha_n} \oplus L_{\alpha_n} \ , \\
&= \cdots = , \\
&= L_{-\alpha_n} \oplus \cdots \oplus L_{-\alpha_1} \oplus H \oplus L_{\alpha_1} \oplus \cdots \oplus L_{\alpha_n} \ , \\
&= \bigoplus_{\alpha\in\Phi^+} L_{-\alpha} \oplus H \oplus \bigoplus_{\alpha\in\Phi^+} L_\alpha \, .
\end{aligned}$$

But this is just the triangular decomposition

$$L = N_- \oplus H \oplus N_+,$$

where

$$N_+ = N(\Delta) = \bigoplus_{\alpha\in\Phi^+} L_\alpha \quad \text{and} \quad N_- = \bigoplus_{\alpha\in\Phi^+} L_{-\alpha}.$$

Appendix B

The Witt Algebra

The Witt algebra is the Lie algebra of derivations of the ring $\mathbb{C}[t, t^{-1}]$, i.e. derivations of the Laurent polynomials, whose basis is given by the vector fields

$$L_n := -t^{n+1}\frac{\partial}{\partial t} \text{ for } n \in \mathbb{N}$$

and Lie bracket $[w, g] := wg - gw$ for w and g in the Witt algebra.

Note that $[L_m, L_n] = (m-n)L_{m+n}$. This is given by,

$$\begin{aligned}
[L_m, L_n] &= -t^{m+1}\frac{\partial}{\partial t}\left(-t^{n+1}\frac{\partial}{\partial t}\right) + t^{n+1}\frac{\partial}{\partial t}\left(-t^{m+1}\frac{\partial}{\partial t}\right)\\
&= -t^{m+1}(n+1)\left(-t^{n}\frac{\partial}{\partial t}\right) + t^{n+1}(m+1)\left(-t^{m}\frac{\partial}{\partial t}\right)\\
&= -(n+1)\left(-t^{m+1+n}\frac{\partial}{\partial t}\right) + (m+1)\left(-t^{n+1+m}\frac{\partial}{\partial t}\right)\\
&= nt^{m+1+n}\frac{\partial}{\partial t} + (-m)t^{n+1+m}\frac{\partial}{\partial t} + (1-1)t^{n+1+m}\frac{\partial}{\partial t}\\
&= nt^{m+1+n}\frac{\partial}{\partial t} + (-m)t^{n+1+m}\frac{\partial}{\partial t}\\
&= -nL_{m+n} + mL_{m+n}\\
&= (m-n)L_{m+n}\,.
\end{aligned}$$

The Witt algebra can be extended uniquely, up to trivial cocyle, by the following bracket (Excercise 2.3)

$$[L_m, L_n] := (m-n)L_{m+n} + \delta_{m,-n}\frac{m^3-m}{12}K.$$

Also, one can show that $Span\{L_{-1}, L_0, L_1\} \cong \mathfrak{sl}_2(\mathbb{C})$.

Appendix C

Calculating the 2-cocycle for $\bar{L}$

Let $a, b \in \bar{L}$, then $\psi_{\bar{L}}(a,b)$ is given by

$$\begin{aligned}
\psi_{\bar{L}}(a,b) &= \psi_{\bar{L}}(t^n \otimes g, t^m \otimes h) \;\text{ for some } g,h \in L\\
&= \kappa_{\bar{L}}(d(t^n)\otimes g, t^m \otimes h)\\
&= \kappa_{\bar{L}}(t\frac{\partial}{\partial t}(t^n)\otimes g, t^m \otimes h)\\
&= \kappa_{\bar{L}}(t(nt^{n-1})\otimes g, t^m \otimes h)\\
&= \kappa_{\bar{L}}(nt^{n}\otimes g, t^m \otimes h)\\
&= n\kappa_{\bar{L}}(t^{n}\otimes g, t^m \otimes h)\\
&= nt^{m+n}|_{t=0}\kappa(g,h)\\
&= \begin{cases} 0 & \text{if } m+n \neq 0\\ n\kappa(g,h) & \text{if } m+n = 0\end{cases}\\
&= \delta_{m,-n}n\kappa(g,h)\,.
\end{aligned}$$

References

[1] J.E. Humphreys, Introduction to Lie Algebras and Representation Theory, 3rd revised printing, Springer Verlag, New York, 1973

[2] V.G. Kac, Infinite dimensional Lie algebras, 3rd edition, Cambridge University Press, Cambridge 1990