SPRING 2021 A643: INTEGRAL EQUATIONS

Contents

1. Week 1

1.1. Introduction.
The aim of this course is to study the equation:

$$\begin{aligned} & u(t) + \int_0^t [Au(s) + B(u(s), u(s)) + F(u(s))] \mathrm{d}s \\ & = u_0 + \int_0^t G(u(s-)) \mathrm{d}W(s) + \int_0^t \int_{E_0} K(u(s-), \xi) \mathrm{d}\widehat{\pi}(s, \xi) + \int_0^t \int_{E \setminus E_0} \mathcal{K}(u(s-), \xi) \mathrm{d}\pi(s, \xi). \end{aligned} \tag{1.1}$$

The LHS of (1.1) represents a deterministic PDE in abstract form. For example, an equation of NSE type with damping:

$$\frac{\mathrm{d}u}{\mathrm{d}t}(x,t) - \Delta u(x,t) + u(x,t) \cdot \nabla u(x,t) + \sqrt{1 + |u(x,t)|^2} u(x,t) \tag{1.2}$$

The RHS of (1.1) represents noise terms:

- $\int_0^t G(u(s-)) \mathrm{d}W(s)$: Stochastic integral w.r.t Wiener process W represents the presence of random forces that are continuous in time.
- $\int_0^t \int_{E \setminus E_0} \mathcal{K}(u(s-), \xi) \mathrm{d}\pi(s, \xi)$: Stochastic integral w.r.t PRM π represents random forces that occur at discrete times.
- $\int_0^t \int_{E_0} K(u(s-), \xi) \mathrm{d}\widehat{\pi}(s, \xi)$: Stochastic integral w.r.t. compensated Poisson Random Measure (PRM) $\widehat{\pi}$ represents "small" random forces that occur at discrete times.

Goal: Is to prove the following theorem (informal statement):

Theorem 1.1. *Under appropriate conditions, for any Wiener process W, PRM π and compensated PRM $\widehat{\pi}$ and any u_0, there exists a unique local solution u to (1.1).*

1.2. Probability concepts.
We consider a set Ω and denote by $\mathcal{F}$ the $\sigma-$algebra of subsets of Ω.

Definition 1.2. *σ-algebra of subsets of Ω is a collection $\mathcal{F}$ of subsets of Ω that contains ϕ and is closed under complement and countable unions. Sets in $\mathcal{F}$ are sometimes called $\mathcal{F}$-measurable.*

Example 1.3. Let $\{F_i\}_{i \in I}$ be a family of $\sigma-$field on Ω. Then $\cap_{i \in I} F_i$ is a σ-field on Ω.
If $A \in \mathcal{P}(\Omega)$, then $\cap_{A \in F} F$ is the σ-field generated by A.

Example 1.4. If (E, d) is a metric space, then we call the σ-field generated by the d-open sets of E the Borel σ-field which we denote by $\mathcal{B}(E)$.

Definition 1.5. *Measurable space is the pair $(\Omega, \mathcal{F})$.*

Definition 1.6. *Probability measure $\mathbf{P}$ on σ-field $\mathcal{F}$ of subsets of Ω is a function from $\mathcal{F}$ to $[0,1]$ such that $\mathbf{P}(\Omega) = 1$ and $\mathbf{P}(\cup_1^\infty A_m) = \sum_1^\infty \mathbf{P}(A_m)$ for each pairwise disjoint sequence $\{A_m; m = 1, 2, ..\}$.*

Definition 1.7. *The triple $(\Omega, \mathcal{F}, \mathbf{P})$ is called a probability space.*

Example 1.8. The product σ-field on $Z_1 \times Z_2 \times ... \times Z_m$ where $(Z_i, \mathcal{Z}_i)$ is a measurable space, is the σ-field generated by $\{\Gamma_1 \times \Gamma_2 ... \times \Gamma_m : \Gamma_i \in \mathcal{Z}_i\}$. We denote it by $\mathcal{Z}_1 \otimes \mathcal{Z}_2 \otimes ... \otimes \mathcal{Z}_m$.

Proposition 1.9. *(Basic properties)*
1. $\mathbf{P}[A] = 1 - \mathbf{P}[A^c]$.
2. If $A_1 \subset A_2$*, then* $\mathbf{P}(A_1) \leq \mathbf{P}(A_2)$.
3. If $(A_n)_{n=1}^{\infty} \subseteq F$*, then* $\mathbf{P}[\cup_n A_n] \leq \sum_{n=1}^{\infty} \mathbf{P}[A_n]$.
4. If $A_n \uparrow A$*,* $\lim_n \mathbf{P}[A_n] = \mathbf{P}[A]$.
5. If $A_n \downarrow A$*,* $\mathbf{P}[A_n] \downarrow \mathbf{P}[A]$.

Lemma 1.10. *(Borel-Cantelli) Let* $(A_n)_{n=1}^{\infty} \subseteq F$ *s.t.* $\sum_n \mathbf{P}[A_n] < \infty$ *then* $\mathbf{P}[\{\omega \in \Omega : \omega \in A_n \text{ for infinitely many } n\}] = 0$.

Proof. (Sketch) $\mathbf{P}(\cap_{k=1}^{\infty} \cup_{n=k}^{\infty} A_n) \leq \mathbf{P}(\cup_{n=k}^{\infty} A_n) \leq \sum_{n=k}^{\infty} \mathbf{P}(A_n) \to 0$ as $k \to \infty$. □

Definition 1.11. *A function* $X : (\Omega, \mathcal{F}) \to (Z, \mathcal{Z})$ *is measurable if* $X^{-1}(A) \in \mathcal{F}$ *for every* $A \in \mathcal{Z}$.

Definition 1.12. *A random variable (r.v.) is a measurable function* X *from a probability space* $(\Omega, \mathcal{F}, \mathbf{P})$ *to some measurable space* $(Z, \mathcal{Z})$.

Notation 1.13. $\{\omega \in \Omega : X(\omega) \in \Gamma\} =: \{X \in \Gamma\} =: \mathbf{X}^{-}(\Gamma)$.

Proposition 1.14. *Let* X *be a r.v. on* $(\Omega, \mathcal{F}, \mathbf{P})$ *with values in a measurable space* $(Z, \mathcal{Z})$*. Then the collection* $\sigma(X) = \{\mathbf{X}^{-}(\Gamma) : \Gamma \in \mathcal{Z}\}$ *is a* σ*-field on* Ω *and* X *is* $\sigma(X)$*-measurable.*

Let $(X_i)_{i \in I}$ be a family of r.v.'s on $(\Omega, \mathcal{F}, \mathbf{P})$. We define $\sigma(X_i : i \in I)$ to be the σ-field generated by $\cup_{i \in I} \sigma(X_i)$. Each X_j is $\sigma(X_i; i \in I)$-measurable.

Law of a r.v. :
If X is a r.v. with values in $(Z, \mathcal{Z})$, then we define a probability measure μ_X on $(Z, \mathcal{Z})$ by

$$\mu_X(\Gamma) = \mathbf{P}(X \in \Gamma) \quad \forall \Gamma \in \mathcal{Z}. \tag{1.3}$$

Example 1.15. Let $\lambda =$ Lebesgue measure on $\mathbb{R}$. For each $\Gamma \in \mathcal{B}(\mathbb{R})$ with $\lambda(\Gamma) < \infty$, we define the uniform distribution on Γ to be the law $\frac{1}{\lambda(\Gamma)} \lambda|_{\Gamma}$. $X\omega) = \omega$ and $Y(\omega) = 1 - \omega$ for $\omega \in [0, 1]$. Then X and Y have uniform distribution on $[0, 1]$.

Definition 1.16. *Let* X_1 *and* X_2 *be two* Z*-valued r.v.s. (possibly on different probability spaces). If* $\mu_{X_1} = \mu_{X_2}$ *then we write*

$$X_1 \overset{\mathcal{D}}{=} X_2$$

and we say that X_1 *and* X_2 *are equal in distribution.*

Proposition 1.17. *Let* X *be a* Z*-valued r.v. on* $(\Omega, \mathcal{F}, \mathbf{P})$ *with law* μ_X*. Then,*
i) For every measurable function $f : Z \to [0, \infty)$ *we have*

$$\int_{\Omega} f(X(\omega))\, \mathrm{d}\mathbf{P}(\omega) = \int_Z f(x)\, \mathrm{d}\mu_X(x) \quad \text{possibly } \infty. \tag{1.4}$$

ii) $f \in L^1(Z, \mathcal{Z}, \mu_X)$ *iff* $f \circ X \in L^1(\Omega, \mathcal{F}, \mathbf{P})$ *and (1.4) holds with both sides finite.*

Proof. (Sketch) if $f = 1_{\Gamma}$ for $\Gamma \in \mathcal{Z}$ then LHS of (1.4) is equal to

$$\int_{\Omega} 1_{X \in \Gamma}\, \mathrm{d}\mathbf{P} = \mathbf{P}(X \in \Gamma) = \mu_X(\Gamma) = \int_Z 1_{\Gamma} \mathrm{d}\mu_X$$

= RHS of (1.4). □

Definition 1.18. *In the same setup as the above proposition we write*

$$\mathbf{E}[f(X)] = \int_\Omega f(X(\omega))\,\mathrm{d}\mathbf{P}(\omega) \tag{1.5}$$

Remark 1.1. *Suppose X takes values in a Hilbert space H (real, separable). If $\mathbf{E}[|X|_H] < \infty$ then there exists a unique $m \in H$ s.t. $\mathbf{E}[(u, X)] = (u, m)$ for all $u \in H$. We write $\mathbf{E}[X] := m$.*

Definition 1.19. *Let X, Y be real valued r.v.'s on $(\Omega, \mathcal{F}, \mathbf{P})$ s.t. $\mathbf{E}[X^2] < \infty$. We define the covariance of X and Y as*

$$cov\ (X, Y) = \mathbf{E}[(X - \mathbf{E}[X])(Y - \mathbf{E}[Y])]. \tag{1.6}$$

The variance of X is var (X)=cov (X, X).

Independence:

Definition 1.20. *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. We say events $A_1, A_2, ...A_m \in \mathcal{F}$ are independent if*

$$\mathbf{P}[\cap_{j\in I} A_j] = \Pi_{j\in I}\mathbf{P}[A_j] \quad \forall I \subseteq \{1, 2, ..., m\}. \tag{1.7}$$

Definition 1.21. *A family $\{\mathcal{F}_i\}_{i\in I}$ of subfields of $\mathcal{F}$ is independent if for every finite $\{i_1, ..., i_m\} \subseteq I$ and every choice of $A_{i_k} \in \mathcal{F}_{i_k}$, $k = 1, ...m$, we have that*

$$\mathbf{P}[\cap_{k=1}^m A_{i_k}] = \Pi_{k=1}^m \mathbf{P}[A_{i_k}]. \tag{1.8}$$

A family $\{X_i\}$ of r.v's on $(\Omega, \mathcal{F}, \mathbf{P})$ is independent if $\{\sigma(X_i)\}_{i\in I}$ is independent in the sense above.

Convergence:

Let $\{X_n\}$ and X be r.v.'s on $(\Omega, \mathcal{F}, \mathbf{P})$ with values in a metric space (E, d).

Definition 1.22. *i) $X_n \to X$ $\mathbf{P}$-a.s. if $\mathbf{P}[\lim_{n\to\infty} X_n = X] = 1$.*
ii) $X_n \to X$ in probability if for every $\varepsilon > 0$ we have $\lim_{n\to\infty} \mathbf{P}[d(X_n, X) > \varepsilon] = 0$.

Proposition 1.23. *i) If $X_n \to X$ $\mathbf{P}$-a.s., then $X_n \to X$ in probability.*
ii) If $X_n \to X$ in probability then there exists a subsequence $(X_{n_k})_{k=1}^\infty$ s.t. $X_{n_k} \to X$ $\mathbf{P}$-a.s.

Definition 1.24. **(1)** *Let $\{\mu_n\}_{n=1}^\infty$ and μ be Borel probability measures (i.e. defined on all open sets of E) on a metric space (E, d). We say that $\{\mu_n\}_{n=1}^\infty$ converges weakly to μ and we write $\mu_n \Rightarrow \mu$ if for every bounded continuous function $f : E \to \mathbb{R}$ we have as*

$$\int_E f\mathrm{d}\mu_n \to \int_E f\mathrm{d}\mu \text{ as } n \to \infty. \tag{1.9}$$

(2) *For E-valued r.v.'s (X_n), X, we say (X_n) converges to X in distribution if $\mu_{X_n} \Rightarrow \mu_X$ and we write $X_n \Rightarrow X$.*

Proposition 1.25. *Let $(X_n)_{n=1}^\infty$, X be E-valued r.v.'s on $(\Omega, \mathcal{F}, \mathbf{P})$:*
i) If $X_n \to X$ in probability then $X_n \Rightarrow X$.
ii) If $\mathbf{P}[X = x_0] = 1$ then $X_n \to x_0$ in probability.

Theorem 1.26. *(Portmanteau theorem) Let $(\mu_n)_{n=1}^{\infty}$ and μ be measures on a metric space E. Then the following are equivalent:*
1. $\mu_n \Rightarrow \mu$
2. $\lim_n \mu_n(A) = \mu(A)$ *for all μ-continuity sets of A.*
3. $\limsup \mu_n(C) \leq \mu(C)$ *for all closed sets C.*
4. $\liminf \mu_n(U) \geq \mu(U)$ *for all open sets U.*
5. $\int f \mathrm{d}\mu_n \to \int f \mathrm{d}\mu$ *for all bounded, Lipschitz continuous f on E.*

Theorem 1.27. *(Skorohod) Let E be a complete and separable metric space. Let $(X_n)_{n=1}^{\infty}$ and X be E-valued r.v.'s on $(\Omega, \mathcal{F}, \mathbf{P})$ such that $X_n \Rightarrow X$. Then there exists a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}})$ and r.v.'s $(\widetilde{X}_n)_{n=1}^{\infty}$ and $\widetilde{X}$ on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}})$ such that*
1. $\widetilde{X}_n \overset{\mathcal{D}}{=} X_n$
2. $\widetilde{X} \overset{\mathcal{D}}{=} X$
3. $\widetilde{X}_n \to \widetilde{X}$ $\widetilde{\mathbf{P}}$*-a.s.*

Proposition 1.28. *(Continuous mapping theorem) Let (E, d), (E', d') be metric spaces. Let $(X_n)_{n=1}^{\infty}$ and X be E-valued r.v.'s such that $X_n \Rightarrow X$. Let $h : E \to E'$ be measurable. Assume that there exists a measurable set $C \subseteq E$ s.t. $\mathbf{P}[X \in C] = 1$ and h is continuous on C. Then, $h(X_n) \Rightarrow h(X)$.*

Proof. (Sketch) Take $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}})$ and $(\widetilde{X}_n)_{n=1}^{\infty}$, $\widetilde{X}$ as in Skorohod s.t. $\widetilde{X}_n \to \widetilde{X}$ $\widetilde{\mathbf{P}}$-a.s. Then $\widetilde{\mathbf{P}}(\widetilde{X} \in C) = 1$. So $h(\widetilde{X}_n) \to h(\widetilde{X})$ $\widetilde{\mathbf{P}}$-a.s. because h is continuous on C. Thus, $h(X_n) =^D h(\widetilde{X}_n) \Rightarrow h(\widetilde{X}) =^D h(X)$. □

Stochastic Processes:

Definition 1.29. *Let $(Z, \mathcal{Z})$ be a measurable space. A family $(X_i)_{i \in I}$ of Z-valued r.v.'s on $(\Omega, \mathcal{F}, \mathbf{P})$ is called a stochastic process. For $\{i_1, ..., i_m\} \subseteq I$, the measures defined on the product space Z^m given by*

$$\mu_{(X_{i_1}, \ldots, X_{i_m})}(S) = \mathbf{P}((X_{i_1}, \ldots X_{i_m}) \in S)$$

are called the finite dimensional distributions (f.d.ds) (marginals) of $(X_i)_{i \in I}$.

Remark 1.2. *In the SPDE context, we look at stochastic processes $(X(t))_{t \geq 0}$ with values in a Banach/ Hilbert space.*

Common Probability Distributions:

Definition 1.30. *A probability density function f of a random variable $X : (\Omega, \mathcal{F}, \mathbf{P}) \to (Z, \mathcal{Z})$ with respect to a measure μ on $(Z, \mathcal{Z})$ is a measurable function with the property*

$$\mathbf{P}(X \in A) = \int_A f \mathrm{d}\mu$$

for any measurable set $A \in \mathcal{Z}$.

Example 1.31. (Poisson Distribution) Let $\lambda > 0$. A discrete random variable is said to have Poisson distribution with parameter λ if

$$\mathbf{P}[X = k] = e^{-\lambda} \frac{\lambda^k}{k!}$$

for $k = 0, 1, 2, ...$ We write $X \sim \text{Pois}(\lambda)$.

$\mathbf{E}[X] = \lambda$ and $\text{Var}[X] = \lambda$. If $X \sim \text{Pois}(\lambda_X)$, $Y \sim \text{Pois}(\lambda_Y)$ are independent then $X + Y \sim \text{Pois}(\lambda_X + \lambda_Y)$.

Example 1.32. (Exponential distribution) Let $\lambda > 0$. We write $X \sim \text{Exp}(\lambda)$ if

$$\mathbf{P}[X > t] = e^{-\lambda t} \quad \forall t \geq 0$$

Probability density function of an exponential distribution is given by: $\lambda e^{-\lambda t}\chi_{(0,\infty)}(t)$. $\mathbf{E}[X] = \frac{1}{\lambda}$, $\text{Var}[X] = \frac{1}{\lambda^2}$.

Example 1.33. (Normal distribution) Let $\mu \in \mathbb{R}, \sigma > 0$. We write $X \sim N(\mu, \sigma^2)$ if

$$\mathbf{P}[X \leq t] = \int_{-\infty}^{t} \frac{1}{2\sqrt{\pi}\sigma} e^{-\frac{(X-\mu)^2}{2\sigma^2}} \mathrm{d}x \quad \forall t \in \mathbb{R}.$$

$\mathbf{E}[X] = \mu$, $\text{Var}[X] = \sigma^2$. And X is called the normal distribution with mean μ and variance σ^2.
If $X \sim N(\mu, \sigma^2)$ then $\frac{X-\mu}{\sigma} \sim N(0, 1)$ (standard normal distribution with mean 0 and variance 1).
If $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ are independent then $X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$ (standard normal distribution).

2. Week 2

2.1. Filtration and stopping times.

Recall from last time:

Definition 2.1. *1) A function $X : (\Omega, \mathcal{F}) \to (Z, \mathcal{Z})$ is measurable if $X^{-1}(A) \in \mathcal{F}$ for every $A \in \mathcal{Z}$.*
2) For $(X_i)_{i\in I}$, family of r.v.'s on $(\Omega, \mathcal{F}, \mathbf{P})$, we define $\sigma(X_i : i \in I)$ to be the σ-field generated by $\cup_{i\in I}\sigma(X_i)$. Each X_j is $\sigma(X_i; i \in I)$-measurable.

Definition 2.2. *A filtration on a probability space is a set $(\mathcal{F}_t)_{t\geq 0}$ of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $s \leq t$.*

We call $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ a filtered probability space.

Example 2.3. Let $(X_t)_{t\geq 0}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Define

$$\mathcal{F}_t^0 = \sigma(X_s, s \leq t).$$

Then $(\mathcal{F}_t^0)_{t\geq 0}$ is a filtration called the natural filtration of the process $(X_t)_{t\geq 0}$.

Definition 2.4. *Let $(X_t)_{t\geq 0}$ be a stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$. We say that $(X_t)_{t\geq 0}$ is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ if for every $t \geq 0$ the random variable X_t is $\mathcal{F}_t$-measurable (measurability defined above).*

Definition 2.5. *Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration on $(\Omega, \mathcal{F}, \mathbf{P})$. A nonnegative function $\tau : \Omega \to [0, \infty]$ is called an $\mathcal{F}_t$-stopping time if*

$$\{\tau \leq t\} \in \mathcal{F}_t \quad \forall t \geq 0.$$

Proposition 2.6. *Let (E, d) be a metric space. Let $(X_t)_{t\geq 0}$ be an E-valued stochastic process defined on $(\Omega, \mathcal{F}, \mathbf{P})$. Suppose that X has continuous paths, i.e., $\forall\, \omega_0 \in \Omega$ the function $t \mapsto X_t(\omega_0)$ is continuous. Then for every closed set $C \subseteq E$ the random variable*

$$\tau_C := \inf\{t \geq 0 : X_t \in C\}$$

is a stopping time with respect to the natural filtration $(\mathcal{F}_t^0)_{t\geq 0}$ of $(X_t)_{t\geq 0}$.

Proof. Since X is continuous (at each sample point) we have $\tau_C \leq t$ if and only if X enters C at or before time t. This occurs if and only if $\inf_{s\in[0,t]} d(X_s, C) = 0$. Therefore,

$$\begin{aligned}\{\tau_C \leq t\} &= \{\inf_{s\in[0,t]} d(X_s, C) = 0\} \\ &= \cap_{n=1}^{\infty} \cup_{s\in[0,t]\cap\mathbb{Q}} \underbrace{\{d(X_s, C) \leq \frac{1}{n}\}}_{\text{belongs to } \mathcal{F}_s^0}\end{aligned}$$

We can express $\{\tau_C \leq t\}$ in terms of countable intersections of countable unions of $\mathcal{F}_t^0$ and hence is in $\mathcal{F}_t^0$. □

Definition 2.7. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ be a filtered probability space. For every $t \geq 0$ we define $\mathcal{F}_{t+} = \cap_{s>t}\mathcal{F}_s$. We say that $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous if $\mathcal{F}_t = \mathcal{F}_{t+}$ for all $t \geq 0$.*

Proposition 2.8. *Suppose $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ is a filtered probability space where $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous. Then a function $\tau : \Omega \to [0, \infty]$ is an $\mathcal{F}_t$-stopping time if and only if*

$$\{\tau < t\} \in \mathcal{F}_t \quad \forall t \geq 0.$$

Proof. For a stopping time τ using the fact that $\mathcal{F}_{s^+} \subseteq \mathcal{F}_t$ for each $s < t$ we obtain

$$\{\tau < t\} = \cup_{n=1}^{\infty} \{\tau \leq t - \frac{1}{n}\} \in \mathcal{F}_t.$$

Conversely, if $\{\tau < t\} \in \mathcal{F}_t$ for each time, then for any $s > t$,

$$\{\tau \leq t\} = \cap_{n=1}^{\infty} \{\tau < (t + \frac{1}{n}) \wedge s\} \in \mathcal{F}_s.$$

Since this is true for all $s > t$ we get $\{\tau \leq t\} \in \mathcal{F}_{t^+}$. □

Proposition 2.9. *Let (E, d) be a metric space. Let $(X_t)_{t\geq 0}$ be an E-valued stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Suppose that $(X_t)_{t\geq 0}$ has right-continuous paths. Consider the natural filtration $(\mathcal{F}_t^0)_{t\geq 0}$ of $(X_t)_{t\geq 0}$. Then for every open set $U \subseteq E$ the random variable*

$$\tau_U := \inf\{t \geq 0 : X_t \in U\}$$

is a stopping time with respect to the filtration $(\mathcal{F}_{t^+}^0)_{t\geq 0}$.

Definition 2.10. *A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ is said to be complete if $\mathcal{F}_0$ contains all of the $\mathbf{P}$-null sets. We say that a filtered probability space satisfies the "usual conditions" (or called normal) if it is both right-continuous and complete.*

Remark 2.1. *Under the "usual conditions" (defined above) assumption, τ_U (U open) is a stopping time with respect to the given filtration.*

2.2. **Martingales.**

Definition 2.11. *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and let H be a real, separable Hilbert space. Let $X : \Omega \to H$ be an integrable H-valued random variable, that is $X_t \in L^1(\Omega, \mathcal{F}, \mathbf{P}; H)$ or $\mathbf{E}|X_t|_H < \infty$. Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. An H-valued random variable $Y : \Omega \to H$ is said to be the conditional expectation of X given $\mathcal{G}$ if*
i) Y is $\mathcal{G}$-measurable.
ii) $\int_G Y \mathrm{d}\mathbf{P} = \int_G X \mathrm{d}\mathbf{P} \quad \forall G \in \mathcal{G}$

Such a random variable exists and is unique ($\mathbf{P}$-a.s.) and we denote it by $Y = \mathbf{E}[X|\mathcal{G}]$.

Remark 2.2. *The integrals in ii) are Bochner integrals. For $X \in L^1(\Omega, \mathcal{F}, \mathbf{P}; H)$ we define $\int_G X \mathrm{d}\mathbf{P}$ by choosing step functions $(X_n)_{n=1}^{\infty}$ converging to X a.s. and in L^1 and setting $\int X \mathrm{d}\mathbf{P} = \lim_{n\to\infty} \int X_n \mathrm{d}\mathbf{P}$.*

Definition 2.12. *(Martingale, continuous time) Let $(X_t)_{t\geq 0}$ be an H-valued stochastic process adapted to $(\mathcal{F}_t)_{t\geq 0}$. We say that X is an $\mathcal{F}_t$-martingale if*
1. $\mathbf{E}(|X_t|_H) < \infty$, $\forall t \geq 0$.
2. $\mathbf{E}[X_t|\mathcal{F}_s] = X_s \quad \forall s < t$.

Definition 2.13. *Consider* $\Pi := \{0 = t_1^n < t_2^n < ... < t_k^n < t_{k+1}^n = T\}$, *a partition of* $[0,T]$. *Then the total variation of a real valued function defined on* $[0,T]$ *is defined to be*

$$TV(f) = \sup_{\Pi} \sum_{1\le k\le n} |f(t_k) - f(t_{k-1})|,$$

where the supremum is taken over all possible partitions Π *of the interval* $[0,T]$ *for all* n. *A real function is said to have bounded variation on* $[0,T]$ *if its total variation is bounded.*

Definition 2.14. *(Semi-martingale) càdlàg = continue à droite, limitée à gauche= right-continuous with left limits.*
Let H *be Hilbert space. A càdlàg ,* H*-valued stochastic process* $(X_t)_{t\ge 0}$ *is called a semi-martingale if* X *can be written in the form*

$$X = M + A,$$

where
i) M *is an* H*-valued martingale s.t.* $\mathbf{E}[|M(t)|_H^2] < \infty$ *for all* $t \ge 0$, *(i.e.* M *is an* L^2*-martingale.)*
ii) A *is an* H*-valued process of finite variation i.e.,* $\forall T \ge 0$ $(A_t(\omega))_{t\in[0,T]}$ *has bounded variation for almost every* $\omega \in \Omega$.

Notation 2.15. *For a càdlàg function* X *we denote,*
1. The left limit of X *at* t *by* $X_{t-} = \lim_{\varepsilon\to 0} X_{t-\varepsilon}$
2. And the size of jumps Δ *by* $\Delta X(t) = X(t) - X(t-)$.

Let $M^2(H) := \{H\text{-valued } L^2\text{-martingales indexed by } [0,\infty)\}$
$M_T^2(H) := \{H\text{-valued } L^2\text{-martingales indexed by } [0,T]\}$ (a Banach space with norm $\mathbf{E}\sup_{t\in[0,T]}|M(t)|_H^2$).

Recall that $a \wedge b$ means the minimum of a and b. reference henceforth "Limit Theorems for Stochastic Processes by Jean Jacod, Albert N. Shiryaev"

Theorem 2.16. *Let* $M, N \in M^2(H)$. *For every* $t \ge 0$ *and every sequence* $(\Pi^n)_{n=1}^\infty$ *of partitions* $\Pi^n := \{0 = t_1^n < t_2^n < ... < t_k^n < ...\}$ *in* $[0,\infty)$ *such that*
1. $\lim_{k\to\infty} t_k^n = \infty$ *for every* n *and,*
2. $\lim_{n\to\infty} \sup_k (t_{k+1}^n - t_k^n) = 0$.
Then the random variables $S_n(t,M,N) := \sum_{k=1}^\infty \langle M(t\wedge t_{k+1}^n) - M(t\wedge t_k^n), N(t\wedge t_{k+1}^n) - N(t\wedge t_k^n)\rangle_H$ *converge in* $L^1(\Omega,\mathcal{F},\mathbf{P})$ *as* $n\to\infty$. *Moreover, the limit does not depend on the choice of the partitions* $(\Pi^n)_{n=1}^\infty$. *As a stochastic process, the limit is* $\mathcal{F}_t$*-adapted and a.s. has right-continuous paths of finite variation.*

Definition 2.17. *For* $M, N \in M^2(H)$ *we define the* **quadratic covariation** *of* M *and* N *as*

$$[M,N]_t := \lim_{n\to\infty} S_n(t,M,N). \quad (L^1\text{- limit})$$

When $M = N$ *we write* $[M]_t := [M,M]_t$ *and call this process the* **quadratic variation** *of* M.

Theorem 2.18. *(Burkholder-Davis-Gundy inequality)* $\forall p \in [1,\infty)$ *there exists* $C = C_p > 0$ *such that for every càdlàg martingale* $M \in M^2(H)$ *with* $M(0) = 0$ *and for every stopping time* τ *we have*

$$\frac{1}{C_p}\mathbf{E}[[M]_\tau^{\frac{p}{2}}] \le \mathbf{E}[\sup_{t\in[0,\tau]} |M(t)|_H^p] \le C_p\mathbf{E}[[M]_\tau^{\frac{p}{2}}].$$

The constant C_p does not depend on M or τ.

Now we discuss additional ways of finding the quadratic variation of a process.

Definition 2.19. *The σ-algebra on $\Omega \times [0,T]$ generated by $A \times (s,t]$ for all $A \in \mathcal{F}_s$, $0 \le s < t \le T$ is called the* **predictable** *σ-field and is denoted by $P_{[0,T]}$. That is,*

$$P_{[0,T]} = \sigma(\{A \times (s,t] : A \in \mathcal{F}_s, 0 \le s < t \le T\}).$$

A function on $\Omega \times [0,T]$ is said to be predictable if it is $P_{[0,T]}$-measurable.

A predictable process is necessarily an adapted one.

Theorem 2.20. *Let $M, N \in M^2(H)$. Then there exists a unique real-valued, predictable process V of finite variation with $V(0) = 0$ a.s., such that $(M_t, N_t)_H - V_t$ is a martingale.*

This is a special case of the Doob-Meyer decomposition.

Definition 2.21. *For $M, N \in M^2(H)$ we define the* **angle bracket** *of M and N as*

$$\langle M, N\rangle_t := V_t,$$

where V is as in Theorem 2.20. When $M = N$ we write $\langle M\rangle_t := \langle M, M\rangle_t$.

Theorem 2.22. *Let $M \in M_T^2(H)$. Then M can be uniquely written as*

$$M = M^c + M^d$$

where $M^c, M^d \in M_T^2(H)$ and M^c is continuous $\mathbf{P}$-a.s and is called the continuous part of M. M^d is the purely discontinuous part.

Theorem 2.23. $[M,N]_t = \langle M^c, N^c\rangle_t + \sum_{s \le t} \langle \Delta M(s), \Delta N(s)\rangle_H$.

Proof. Show that the subspace of continuous martingales in $M_T^2(H)$ is closed. Since $M_T^2(H)$ is a Hilbert space, the conclusion follows from the projection theorem. ☐

Example 2.24. For Brownian motion, we have

$$[M]_t = \langle M^c\rangle_t + \underbrace{\sum_{s \le t} |\Delta M(s)|_H^2}_{\text{=0 because no jump discontinuities}}.$$

3. Week 3

Recall:

Definition 3.1. *(1) A filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ *is said to be complete (or is said to have a complete filtration) if* $\mathcal{F}_0$ *contains all of the sets* $A \in \mathcal{F}$ *such that* $\mathbf{P}(A) = 0$ *and right-continuous if* $\cap_{s>t}\mathcal{F}_s := \mathcal{F}_{t^+} = \mathcal{F}_t$ *for all* $t \geq 0$*. We say that a filtered probability space satisfies the "usual conditions" (or called normal) if it is both right-continuous and complete. (Even though Da Prato or Rockner (pg 16) doesn't assume it, in most literatures the underlying probability space* $(\Omega, \mathcal{F}, \mathbf{P})$ *is also assumed to be complete (that is, it contains all subsets of 0 probability i.e. the set* $A \subseteq \Omega : A \subseteq B$ *for some* $B \in \mathcal{F}$ *s.t.* $\mathbf{P}(B) = 0$ *also belongs to* $\mathcal{F}$*)).*

(2) Natural filtration is given by $\mathcal{F}_t^0 = \sigma(X_s, s \leq t)$*. (* $\mathcal{F}_t^0$*, is the* σ*-algebra of events occuring up to time t i.e. the "past events up to t".)*

(3) Predictable σ*- field is given by* $P_{[0,T]} = \sigma(\{A \times (s,t] : A \in \mathcal{F}_s, 0 \leq s < t \leq T\})$*. A predictable process is a function defined on* $\Omega \times [0,T]$ *which is* $P_{[0,T]}$*-measurable.*

3.1. **Lévy Processes.** Examples in $1D$:

Definition 3.2. *A stochastic process* $(B_t)_{t\geq 0}$ *is said to be a Brownian motion in* $1D$ *if it is* $\mathbb{R}$*-valued and*

(1) $B_0 = 0$*,*
(2) B_t *is continuous a.s., i.e.* $\mathbf{P}[\{\omega : t \mapsto B_t(\omega) \text{ is continuous}\}] = 1$*,*
(3) B has independent increments: $\forall\, 0 \leq t_1 < t_2 < \ldots < t_k$ *the r.v.s* $B_{t_2} - B_{t_1}, \ldots, B_{t_k} - B_{t_{k-1}}$ *are independent.*
(4) $\forall\, t \geq s \geq 0, B_t - B_s \sim N(0, t-s)$ *(in fact are stationary* $\sim B_{t-s}$*).*

Properties of Brownian motion sample paths

1) Nowhere differentiable
2) Hölder continuous of order α for all $\alpha \in (0, \frac{1}{2})$.
3) No bounded total variation.
4) Yet it has bounded quadratic variation. Ex. QV of BM on an interval $[a,b]$ is $b-a$.

3.2. **Lévy Processes in ∞-dim.**

Definition 3.3. *Let U be a Hilbert space. A U-valued stochastic process* $(L(t))_{t\geq 0}$ *is called a Lévy process if*
i. $L(0) = 0$*,*
ii. L has independent increments,
iii. L has stationary increments,
iv. Stochastic continuity: $\forall\, t_0 \geq 0$*,* $L(t) \to L(t_0)$ *in probability as* $t \to t_0$ *i.e.* $\forall\, \varepsilon > 0$*,*

$$\lim_{t\downarrow t_0} \mathbf{P}(|L(t) - L(t_0)|_U > \varepsilon) = 0.$$

Brownian motion and Poisson process are $\mathbb{R}$-valued Lévy process.

A Wiener process is a U-valued, integrable, mean-zero Lévy process with a.s. continuous paths i.e. $\forall x \in U$, $\mathbf{E}\langle W(t), x\rangle_U = 0$ for all $t \geq 0$ and with probability 1 the function $t \mapsto W(t)$ is continuous from $[0, \infty)$ to U (in other words $\mathbf{P}[\{\omega : t \mapsto W_t(\omega) \text{ is continuous}\}] = 1$). Hence the only possible difference Lévy processes and Wiener processes is the possible occurrence of discontinuities in Lévy processes. However we have the following result:

Theorem 3.4. *Every Lévy process has a càdlàg modification i.e. there exists a U-valued Lévy process $\widetilde{L}$ s.t.*
i. $\widetilde{L}$ has càdlàg sample paths a.s.
ii. $\forall t \geq 0$ $\mathbf{P}[\widetilde{L}(t) = L(t)] = 1$.

3.3. **Wiener Process.** (It can be seen that a Wiener process W is a martingale with respect to its natural filtration $\mathcal{F}_t^0 := \sigma(W(s) : s \leq t)$ because W has independent increments. Even if the natural filtration is not complete and right continuous, it can be enlarged to $\widetilde{\mathcal{F}}_t^0 = \cap_{s>t}\sigma(\mathcal{F}_s^0 \cup \mathcal{N})$ minimally that satisfies the usual conditions so that W is $\widetilde{\mathcal{F}}_t^0$-adapted. Then the Lévy theorem states that any Wiener process is a Brownian motion process. And thus the terminologies can be interchanged. The following theorem guarantees that all the different definitions coincide.)

Theorem 3.5. *Let W be a U-valued Wiener process. Then*
1) $\mathbf{E}|W(t)|_U^2 < \infty$ for all $t \geq 0$.
2) (Gaussian) $\forall t_1, ..., t_n \geq 0$, $\forall u_1, ..., u_n \in U$ the random vector $(\langle W(t_1), u_1\rangle_U, ..., \langle W(t_n), u_n\rangle_U)$ has multivariate normal distribution on $\mathbb{R}^n$, with mean zero. That is, $\forall \Gamma \in \mathcal{B}(\mathbb{R}^n)$,

$$\mathbf{P}((\langle W(t_1), u_1\rangle_U, ..., \langle W(t_n), u_n\rangle_U) \in \Gamma) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} \int_\Gamma e^{-\frac{1}{2}y^T\Sigma^{-1}y} \mathrm{d}y$$

$y = (y_1, ..., y_n)$, where Σ is the $n \times n$ covariance matrix given by

$$\Sigma_{ij} = \mathbf{E}[\langle W(t_i), u_i\rangle_U \langle W(t_j), u_j\rangle_U]$$

Definition 3.6. *Let $Q : U \to U$ be a bounded, linear, symmetric, positive, trace class on U then we say $Q \in L_1^+(U)$.*
(Symmetric: $\langle Qx, y\rangle_U = \langle x, Qy\rangle_U$, $\forall x, y \in U$.
Positive: $\forall x \in U$, $\langle Qx, x\rangle \geq 0$.
Trace class: There exists sequences $(a_k), (b_k) \in U$ s.t. $Qu = \sum_k \langle a_k, u\rangle b_k$, for all $u \in U$ and $\sum_k |a_k|_U |b_k|_U < \infty$.)

Representation of the Wiener Process: Let $Q \in L_1^+(U)$ (then its compact). By the spectral theorem there exists an orthonormal basis (ONB) $(u_n)_1^\infty$ of U consisting of eigenvectors of Q with corresponding (nonnegative) eigenvalues $(\gamma_n)_1^\infty$. Let $(\beta_n)_1^\infty$ be a sequence of independent identically distributed (i.i.d.) standard real-valued Brownian motions. Define W by

$$W = \sum_{n=1}^{\infty} \sqrt{\gamma_n}\beta_n(t)u_n. \tag{3.1}$$

Proposition 3.7. *The sum in (3.1) converges in $L^2(\Omega, C([0, T], U))$ for all $T \geq 0$.*

Proof. The Burkholder-Davis-Gundy inequality gives us:

$$\begin{aligned}\mathbf{E}[\sup_{t\in[0,T]} |\sum_{n=k}^{m} \gamma_n\beta_n(t)u_n|_U^2] \leq & C\mathbf{E}([\sum_{n=k}^{m} \gamma_n\beta_n(t)u_n]_T)(\text{ quadratic variation}) \\ &= C\sum_{n,j=k}^{m} \sqrt{\gamma_n\gamma_j}\mathbf{E}([\beta_n u_n, \beta_j u_j]_T) \\ &= C\sum_{n,j=k}^{m} \sqrt{\gamma_n\gamma_j}\delta_{nj}\mathbf{E}[\beta_n, \beta_j]_T \\ &= CT\sum_{n=k}^{m} \gamma_n \to 0 \text{ as } k, m \to \infty.\end{aligned}$$

□

Remark 3.1. (3.1) *also converges a.s. in* $C([0,T];U)$ *for all* $T \geq 0$ *(Da Prato-Zabczyk). The process* W *in* (3.1) *is a Lévy process and its sample paths are continuous a.s.*

Remark 3.2. *For any Wiener process* W *with values in* U *there exists a* $Q \in L_1^+(U)$*, some i.i.d. BMs* $\{\beta_n\}_{n=1}^\infty$*, such that* (3.1) *holds. This* Q *is called the covariance operator of* W*.*

Examples of a Lévy process with jump discontinuities:

Recall: (Poisson Distribution) Let $\lambda > 0$. A discrete random variable is said to have Poisson distribution with parameter λ if

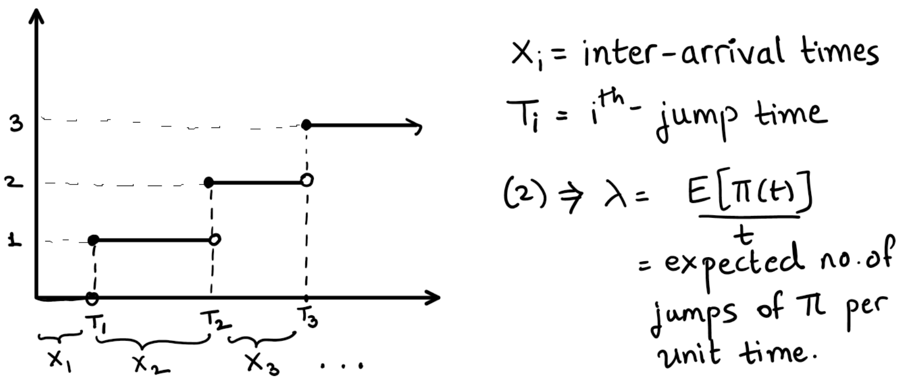
$$\mathbf{P}[X = k] = e^{-\lambda}\frac{\lambda^k}{k!}$$

for $k = 0, 1, 2, ...$ We write $X \sim \text{Pois}(\lambda)$.

Definition 3.8. *A real-valued, right-continuous stochastic process* $(\pi(t))_{t\geq 0}$ *is called a Poisson process if*

(1) $\pi(0) = 0$,
(2) π *has independent increments,*
(3) There exists $\lambda > 0$ *s.t.* $\pi(t) - \pi(s) \sim Pois(\lambda(t-s))$ *for all* $t > s > 0$*.* λ *is called the intensity measure of* $(\pi(t))_{t\geq 0}$*.*

Proposition 3.9. *Let* $(\pi(t))_{t\geq 0}$ *be a Poisson process with intensity* λ*. Then,*
(1) π *has stationary increments i.e.* $\pi(t) - \pi(s) \overset{\mathcal{D}}{=} \pi(t-s)$*.*
(2) $\mathbf{E}[\pi(t)] = \lambda t$ *for all* $t \geq 0$*,*
(3) $\mathbf{P}[\Delta\pi(t) \in \{0,1\}, \forall t \geq 0] = 1$ *where recall that* $\Delta\pi(t) = \pi(t) - \pi(t-)$*. (jump size=1 a.s.)*



Recall that a r.v. X has an exponential distribution at rate λ, i.e. $X \sim \exp(\lambda)$, if $X \geq 0$ and $\mathbf{P}(X > x) = e^{-\lambda x}$. An important property of the exp distribution is the memory-less property $\mathbf{P}(X - y > x | X > y) = \mathbf{P}(X > x)$ for $x, y \geq 0$. (The converse is also true.)

Proposition 3.10. *Let π be a Poisson process with intensity λ and inter arrival times $(X_n)_{n=1}^{\infty}$. Then $(X_n)_{n=1}^{\infty}$ are independent and identically distributed (i.i.ds) $\exp(\lambda)$ r.v's.*

Proof. The proof uses induction and follows the scheme:

$$\mathbf{P}(X_1 > t) = \mathbf{P}(\pi(t) = 0) = e^{-\lambda t}$$

Then let $t = x_{n+1}$

$$\begin{aligned}
&\mathbf{P}[X_{n+1} > t | X_1 = x_1, ..., X_n = x_n] \\
&= \mathbf{P}[\pi(\sum_{k=1}^{n+1} x_k) - \pi(\sum_{k=1}^{n} x_k) = 0 | X_1 = x_1, ..., X_n = x_n] \\
&= \mathbf{P}[\pi(\sum_{k=1}^{n+1} x_k) - \pi(\sum_{k=1}^{n} x_k) = 0] \quad \text{independent increments} \\
&= \mathbf{P}[\pi(\sum_{k=1}^{n+1} x_k - \sum_{k=1}^{n} x_k) = 0] \quad \text{stationary increments} \\
&= e^{-\lambda t} \quad \text{Poisson distribution.}
\end{aligned}$$

□

Properties of Poisson processes:

Spatial homogeneity: Conditional on the event $\pi(t) - \pi(s) = n$ for $t > s \geq 0$, the n jumps in $(s, t]$ are uniformly distributed in $(s, t]$.

3.4. **Compound Poisson Process.** (The Hilbert space-valued generalization of the Poisson process.)

Definition 3.11. *Let μ be a finite Borel measure on U s.t. $\mu(\{0\}) = 0$. A compound Poisson process (CPP) with jump intensity measure (or Lévy measure) μ is a Lévy process P with càdlàg U-valued sample paths such that*

$$\mathbf{P}[P(t) \in \Gamma] = e^{-\mu(U)t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \mu^{*k}(\Gamma) \quad \forall\, t \geq 0, \Gamma \in \mathcal{B}(U) \tag{3.2}$$

where $\mu^{*j} := \underbrace{\mu * \mu ... * \mu}_{j \text{ times}}$, $\mu^{*0} := \delta_0$. Here we use δ_0 to denote the probability measure concentrated at the point $0 \in U$.
Recall (convolution): If ν_1 and ν_2 are finite positive Borel measures on U then their convolution is the measure given by: $(\nu_1 * \nu_2)(\Gamma) := \int_U \nu_1(\Gamma - y) \mathrm{d}\nu_2(y)$ for all $\Gamma \in \mathcal{B}(U)$ where $\Gamma - y = \{x - y : x \in \Gamma\}$.

Theorem 3.12. *1) Let $\{Z_n\}_{n=1}^{\infty}$ be i.i.d. U-valued r.v.s with law $\frac{1}{\mu(U)}\mu$ and let π be a Poisson process with intensity $\lambda = \mu(U)$ and π independent of $\{Z_n\}_{n=1}^{\infty}$. Then*

$$P(t) := \sum_{n=1}^{\pi(t)} Z_n \tag{3.3}$$

is a CPP with jump intensity measure μ.
2) Every CPP has the form in (3.3). (In the proof one defines $\pi(t) := \#\{s \leq t : \Delta P(s) \neq 0\}$, and then $Z_k = P(T_k) - P(T_{k-1})$ have the desired properties where T_k is the k-th jump times.)

In fact let L be a U-valued Lévy process. Let $A \in \mathcal{B}(U)$ s.t. $0 \notin \bar{A}$ (A is seperated from 0). Define the real valued process (called the jump measure of L)

$$\pi([0,t], A) := \sum_{s \geq 0, \Delta L(s) \in A} \delta_{(s, \Delta L(s))} = \sum_{s \in (0,t]} \chi_A(\Delta L(s)) = \#\{s \in (0,t] : \Delta L(s) \in A\}$$

Remark 3.3. *Since $0 \notin \bar{A}$ and L is càdlàg a.s. it follows that $\pi([0,t], A) < \infty$ a.s.*

Fact: For a fixed A, $(\pi_A(t) := \pi([0,t], A))_{t \geq 0}$ is a Poisson process.

Let $\nu(A)$ denote the intensity of π_A (so $\nu_A \in (0, \infty)$),

$$\nu(A) = \frac{1}{t} \mathbf{E}[\pi_A(t)] = \mathbf{E}[\pi_A(1)]$$

(used to define $\nu(A)$ because it makes sense even if $0 \in \bar{A}$)
ν is a Borel measure on $U \setminus \{0\}$. (Use Tonelli's theorem to show that ν is countably additive.)

Definition 3.13. *We call ν the Lévy measure of L.*

Remark 3.4. *If L is a CPP then ν is equal to its jump intensity measure μ*

Definition 3.14. *Let $\mathcal{M}$ be the collection of non-negative, possibly infinite, $\bar{\mathbb{N}}$-valued measures on $(E, \mathcal{E})$. A Poisson random measure on $(E, \mathcal{E})$ with intensity measure λ is a $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$-valued random variable π defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with the property that for all pairwise disjoint sets $\Gamma_1, .., \Gamma_n \in \mathcal{E}$ the $\bar{\mathbb{N}}$-valued random variables $\pi(\Gamma_1), ..., \pi(\Gamma_n)$ are independent Poisson* *random variables with parameters* $\lambda(\Gamma_1), ..., \lambda(\Gamma_n)$. *We call* $\widehat{\pi} := \pi - \lambda$ *the compensated Poisson random measure.*

Relationship to SPDE (1.1)**:** $\int_{[0,t]}\int_{E\setminus E_0} \mathscr{K}(u(s-),\xi)\mathrm{d}\pi(s,\xi)$ is defined based on a CPP where $E \setminus E_0$ is the support of μ.
π is measure-valued r.v. defined by

$$\pi = \sum_{s\geq 0, \Delta P(s)\neq 0} \delta_{(s,\Delta P(s))} = \sum_{n=1}^{\infty} \delta_{(T_n, Z_n)}$$

where $\{T_n\}_1^\infty$ are the jump times of P and $Z_n := \Delta P(T_n)$.

$$\begin{aligned}\int_{[0,T]}\int_{E\setminus E_0} \mathscr{K}(u(s-),\xi)\mathrm{d}\pi(s,\xi) &= \sum_{s\geq 0, \Delta P(s)\neq 0} \mathscr{K}(u(s-),\Delta P(s)) \\ &= \sum_{n=1}^{\pi(t)} \mathscr{K}(u(T_n-), Z_n) \text{ (finitely many terms a.s.)}\end{aligned}$$

3.5. **Compensated CPP.**

Proposition 3.15. *Let* P *be a* U*-valued CPP with Lévy (jump intensity) measure* μ*. Then,*

(1) $\mathbf{E}|P(t)|_U < \infty$, $\forall\, t$ *if and only if* $\int_U |y|_U \mathrm{d}\mu(y) < \infty$ *and then* $\mathbf{E}[P(t)] = t\int_U y\, \mathrm{d}\mu(y)$ *holds for all* $t \geq 0$.

(2) $\mathbf{E}[|P(t)|_U^2] < \infty$, $\forall\, t \geq 0$ *if and only if* $\int_U |y|_U^2\, \mathrm{d}\mu(y) < \infty$.

Proof. For $\lambda = \mu(U)$ we have

$$\begin{aligned}\mathbf{E}P(t) &= \sum_{k=1}^{\infty} \mathbf{E}\left[\left(\sum_{i=1}^{k} Z_i\right)\chi_{\{\pi(t)=k\}}\right] \\ &= \sum_{k=1}^{\infty} \int_U \cdots \int_U \left(\sum_{i=1}^{k} y_i\right)(\lambda)^{-k}\mathrm{d}\mu(y_1)...\mu(y_k)\, \underbrace{e^{-\lambda t}\frac{(\lambda t)^k}{k!}}_{\text{law of } \pi} \\ &= \sum_{k=1}^{\infty} k\lambda^{k-1}e^{-\lambda t}\frac{t^k}{k!}\int_U y\mathrm{d}\mu(y) \\ &= t\int_U y\mathrm{d}\mu(y).\end{aligned}$$

□

Definition 3.16. *When (1) holds, we define*

$$\widehat{P}(t) := P(t) - \mathbf{E}[P(t)] = P(t) - t\int_U y\,\mathrm{d}\mu(y).$$

We call $\widehat{P}$ *a compensated CPP.*

CCPP is a mean-zero Lévy process, with jump discontinuities.

Remark 3.5. *We will define the term* $\int_{[0,T]}\int_{E_0} K(u(s-),\xi)\mathrm{d}\widehat{\pi}(s,\xi)$ *based on a compensated CPP (Here* π *is a Poisson random measure and* $\widehat{\pi}$ *is the compensated Poisson random measure).*

4. Week 4

Recall: For an integrable U-valued compound Poisson process (CPP) P with Lévy (jump intensity) measure μ, the process given by $\widehat{P}(t) := P(t) - \mathbf{E}[P(t)] = P(t) - t\int_U y\,\mathrm{d}\mu(y)$ is called the compensated CPP.

Lemma 4.1. *(PZ 3.25/4.49) Let L be a mean-zero Lévy process, then L is a martingale w.r.t. its natural filtration (This comes from the independent increments conditions). So, Wiener processes and compensated CPPs are martingales w.r.t their natural filtration. (A Poisson process on the other hand is not martingale. Sum of two compensated CPPs may not be martingale either.)*

Lemma 4.2. *Let L be a Lévy process with Lévy measure ν. Then*

(1) $\int_U (|y|_U^2 \wedge 1)\mathrm{d}\nu(y) < \infty$

(2) If $A \in \mathcal{B}(U)$ with $0 \notin \bar{A}$ then

$$L_A(t) := \sum_{s\in(0,t]} \Delta L(s)\chi_A(\Delta L(s))$$

is a CPP with Lévy measure $\nu|_A$.

4.1. Lévy -Khinchin decomposition.

(Which basically says that a Lévy process is a sum of a deterministic linear growth term, a Wiener process, a compound Poisson process and compensated compound Poisson processes.)

Theorem 4.3. *(L-K) Let L be a U-valued Lévy process with Lévy measure ν. Given $\{r_n\}_{n=1}^\infty$, $r_n \downarrow 0$, define $A_0 := \{y \in U : |y|_U \geq r_0\}$ and $A_n := \{y \in U : r_{n+1} \leq |y|_U < r_n\}$. Then the following statements hold.*

(1) The compound Poisson processes $(L_{A_n})_{n=0}^\infty$ are independent.

(2) There exists $a \in U$ and a Wiener process W that is independent of $(L_{A_n})_{n=0}^\infty$ such that

$$L(t) = at + W(t) + L_{A_0}(t) + \sum_{n=1}^\infty \widehat{L}_{A_n}(t) \tag{4.1}$$

and, with probability 1, the series on the right-hand side of (4.1) converges uniformly on compact subsets of $[0,\infty)$.

Remark 4.1. *1) Informally, integration with respect to a Lévy process L decomposes as $\mathrm{d}L = a\mathrm{d}t + \mathrm{d}W + \mathrm{d}\widehat{\pi} + \mathrm{d}\pi$, where π (called the jump measure) is the Poisson random measure given by $\sum_{t>0}\delta_{(t,\Delta L(t))}$.*
2) This decomposition is needed to identify the jumps and quadratic variation of stochastic integrals needed to apply Ito's formula to get a priori estimates.

4.2. Stochastic integration.

Recall from last class, we introduced the covariance operator $Q \in L_1^+(U)$ (ONB u_n and e-values γ_n) associated with a Wiener process. Now for $x \in U$ we define:

$$Q^{\frac{1}{2}}x = \sum_n \langle x, u_n\rangle_U \gamma_n^{\frac{1}{2}} u_n.$$

Like Q, $Q^{\frac{1}{2}}$ is Hilbert-Schmidt and the space $U_0 := Q^{\frac{1}{2}}(U)$ is Hilbert with inner product $(x,y)_{U_0} = \langle Q^{-\frac{1}{2}}x, Q^{-\frac{1}{2}}y\rangle_U$. Here $Q^{-\frac{1}{2}} : U_0 \to \ker(Q^{\frac{1}{2}})^\perp$ is the pseudo-inverse of $Q^{\frac{1}{2}}$ (if $Q^{\frac{1}{2}}$ is not 1-1).

Notes:

(1) A linear operator $T \in L(U,H)$ is called Hilbert-Schmidt if $\sum_k |T\widetilde{e}_k|_H^2 < \infty$ for any (or equivalently for a certain) orthonormal basis $\{\widetilde{e}_k\}$ of U.
The norm $\|T\|_{L_2(U,H)} = (\sum_{n=1}^{\infty} \langle T\widetilde{e}_n, T\widetilde{e}_n\rangle_H)^{\frac{1}{2}}$ does not depend on the choice of the basis. The space of Hilbert Schmidt operators from U to H is a Hilbert space and is denoted by $L_2(U,H)$.
(2) Pseudo-inverse of an operator $T \in L(U,H)$ is defined as

$$T^{-1} = (T|_{\ker(T)^\perp})^{-1} : T(U) \mapsto \ker(T)^\perp.$$

4.2.1. *Wiener stochastic integral.*

Definition 4.4. *Suppose a Lévy process L on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ satisfies:*

(1) L is $\mathcal{F}_t$-adapted.
(2) $L(t) - L(s)$ is independent of $\mathcal{F}_s$, for all $t \geq s \geq 0$.

(Often called $\mathcal{F}_t$-Lévy process.)

Definition 4.5. *Let W be an $\mathcal{F}_t$-Wiener process. Covariance operator Q of W is a linear operator on U defined by*

$$\langle Qx, y\rangle_U := \mathbf{E}[\langle W(1), x\rangle_U \langle W(1), y\rangle_U] \quad \forall\, x, y \in U \tag{4.2}$$

Note: Observe that we do have that $Q \in L_1^+(U)$:

(1) (bounded) $|\text{RHS}| \leq \mathbf{E}[|W(1)|^2]|x|_U|y|_U < \infty$
(2) (self-adjoint) $\langle Qx, y\rangle_U = \langle x, Qy\rangle_U$
(3) (positive) $\langle Qx, x\rangle_U \geq 0$
(4) (trace class) Tr Q $= \sum_1^\infty \langle Qu_n, u_n\rangle_U = \mathbf{E}[|W(1)|^2] < \infty$

In fact we have: For any U-valued square-integrable, mean-zero $\mathcal{F}_t$-Lévy process L there exists a $Q \in L_1^+(U)$ s.t. $\mathbf{E}[\langle L(t), x\rangle_U \langle L(s), y\rangle_U] = (t \wedge s)\langle Qx, y\rangle_U$ where $TrQ = \frac{\mathbf{E}[|L(t)|_U^2]}{t}$

Remark 4.2. *Hence Q determines the law of Wiener process.*

Lemma 4.6. *(1)* $\forall\, x, y \in U,\ 0 < s \leq t$
$\mathbf{E}[\langle W(t) - W(s), x\rangle_U \langle W(t) - W(s), y\rangle_U | \mathcal{F}_s] = (t-s)\langle Qx, y\rangle_U$, $\mathbf{P}$*-a.s.*
(2) $\forall\, x, y \in U,\ 0 < s \leq t \leq u \leq v$
$\mathbf{E}[\langle W(t) - W(s), x\rangle_U \langle W(v) - W(u), y\rangle_U | \mathcal{F}_u] = 0$, $\mathbf{P}$*-a.s.*

Proof. (1) Since $W(t) - W(s)$ is independent of $\mathcal{F}_s$

$$\begin{aligned} \text{LHS} &= \mathbf{E}[\text{ without condition }] \\ &= \mathbf{E}[\langle W(t-s), x\rangle_U \langle W(t-s), y\rangle_U] \\ &= (t-s)\langle Qx, y\rangle_U \end{aligned}$$

(2) If Y is $\mathcal{G}$-measurable $\mathbf{E}[XY|\mathcal{G}] = Y\mathbf{E}[X|G]$. So since $\langle W(t)-W(s), x\rangle_U$ is $\mathcal{F}_u$-measurable:

$$\begin{aligned}\text{LHS} &= \langle W(t)-W(s), x\rangle_U \mathbf{E}[\langle W(v)-W(u), y\rangle_U|\mathcal{F}_u] \\ &= \langle W(t)-W(s), x\rangle_U \mathbf{E}[\langle W(v)-W(u), y\rangle_U] \\ &= 0.\end{aligned}$$

□

Define stochastic integral:

Fix another separable Hilbert space H. Integrand will be $L(U,H)$-valued (space of linear bounded operators from U to H).

Definition 4.7. *(Simple process) (PZ def 8.5) An $L(U,H)$ valued stochastic process Ψ is said to be simple if there exists a sequence of non-negative numbers $t_0 = 0 < t_1 < ... < t_m$, a sequence of operators $\Phi_j \in L(U,H)$, $j = 1, 2...m$ and a sequence of events $A_j \in \mathcal{F}_{t_j}$, $j = 0, 1, ..., m-1$, such that for $s \geq 0$*

$$\Psi(\omega, s) = \sum_{j=0}^{m-1} \mathbb{1}_{A_j}(\omega)\mathbb{1}_{(t_j,t_{j+1}]}(s)\Phi_j$$

We denote by $S(U,H)$ the space of all simple processes from U to H.

(Note: The space $S(U,H)$ depends on $(\mathcal{F}_t)_{t\geq 0}$. The condition $A_j \in \mathcal{F}_{t_j}$ implies that Ψ is a predictable process.)

Definition 4.8. *For a simple process $\Psi \in S(U,H)$, we define the stochastic integral w.r.t W by*

$$I_t^W(\Psi) := \int_0^t \Psi(s)\mathrm{d}W(s) := \sum_{j=0}^{m-1} \mathbb{1}_{A_j}\Phi_j(W(t_{j+1}\wedge t) - W(t_j \wedge t))$$

$I_t^W(\Psi)$ is H-valued continuous ($\mathbf{P}$-a.s.) stochastic process because W is continuous.

Proposition 4.9. *(Itô isometry) For a simple process $\Psi \in S(U,H)$, $\forall t \geq 0$ we have*

$$\mathbf{E}[|I_t^W(\Psi)|_H^2] = \mathbf{E}[|\int_0^t \Psi(s)\mathrm{d}W(s)|_H^2] = \mathbf{E}\int_0^t \|\Psi(s)Q^{\frac{1}{2}}\|_{L_2(U,H)}^2 \mathrm{d}s = \mathbf{E}\int_0^t \|\Psi(s)\|_{L_2(U_0,H)}^2 \mathrm{d}s \tag{4.3}$$

where Q is the covariance operator of W.

(Note: The last equality comes from the fact that $\{Q^{\frac{1}{2}}\tilde{e}_n\}_1^\infty$ form an orthonormal basis of U_0 where $\{\tilde{e}_n\}_1^\infty$ is an ONB of U and that the Hilbert-Schmidt norm does not depend on the ONB.)

Proof. WLOG assume $0 < t_1 < ... < t_m \leq t$.

$$\begin{aligned}\mathbf{E}[|I_t^W(\Psi)|_H^2] &= \mathbf{E}[|\sum_{j=0}^{m-1} \mathbb{1}_{A_j}\Phi_j(W(t_{j+1}) - W(t_j))|_H^2] \\ &= \sum_{j,k=0}^{m-1} \mathbf{E}[\mathbb{1}_{A_j}\mathbb{1}_{A_k}\langle \Phi_j(W(t_{j+1}) - W(t_j)), \Phi_k(W(t_{k+1}) - W(t_k))\rangle_H]\end{aligned}$$

Using Parseval's identity we obtain: (Here $\{e_n\}$ is the ONB of H)

$$= \sum_{j,k=0}^{m-1} \mathbf{E}[\mathbb{1}_{A_j\cap A_k} \sum_{n=1}^{\infty} \langle \Phi_j(W(t_{j+1}) - W(t_j)), e_n\rangle_H \langle \Phi_k(W(t_{k+1}) - W(t_k)), e_n\rangle_H]$$

$$\text{(Fubini)} = \sum_{j,k=0}^{m-1} \sum_{n=1}^{\infty} \mathbf{E}[\mathbb{1}_{A_j\cap A_k} \langle W(t_{j+1}) - W(t_j), \Phi_j^* e_n\rangle_U \langle W(t_{k+1}) - W(t_k), \Phi_k^* e_n\rangle_U].$$

When $k > j$ since $\mathbb{1}_{A_j\cap A_k}$ is $\mathcal{F}_{t_k}$ measurable we have (from tower property)

$$\begin{aligned} &\mathbf{E}[\mathbb{1}_{A_j\cap A_k} \langle W(t_{j+1}) - W(t_j), \Phi_j^* e_n\rangle_U \langle W(t_{k+1}) - W(t_k), \Phi_k^* e_n\rangle_U] \\ &= \mathbf{E}[\mathbb{1}_{A_j\cap A_k} \mathbf{E}[\langle W(t_{j+1}) - W(t_j), \Phi_j^* e_n\rangle_U \langle W(t_{k+1}) - W(t_k), \Phi_k^* e_n\rangle_U | \mathcal{F}_{t_k}]] \\ \text{(Lemma 6.1)} &= 0 \end{aligned}$$

and so

$$\begin{aligned} \mathbf{E}[|I_t^W(\Psi)|_H^2] &= \sum_{j=0}^{m-1} \sum_{n=1}^{\infty} \mathbf{E}[\mathbb{1}_{A_j} \langle W(t_{j+1}) - W(t_j), \Phi_j^* e_n\rangle_U^2] \\ &= \sum_{j=0}^{m-1} \sum_{n=1}^{\infty} \mathbf{E}[\mathbb{1}_{A_j} \mathbf{E}[\langle W(t_{j+1}) - W(t_j), \Phi_j^* e_n\rangle_U^2 | \mathcal{F}_{t_j}]] \\ \text{(Lemma 6.1)} &= \sum_{j=0}^{m-1} \sum_{n=1}^{\infty} \mathbf{E}[\mathbb{1}_{A_j} (t_{j+1} - t_j) \langle Q\Phi_j^* e_n, \Phi_j^* e_n\rangle_U] \\ &= \sum_{j=0}^{m-1} \sum_{n=1}^{\infty} \mathbf{P}(A_j)(t_{j+1} - t_j) \langle Q\Phi_j^* e_n, \Phi_j^* e_n\rangle_U \\ &= \sum_{j=0}^{m-1} \mathbf{P}(A_j)(t_{j+1} - t_j) \sum_{n=1}^{\infty} |Q^{\frac{1}{2}} \Phi_j^* e_n|_U^2 \\ &= \sum_{j=0}^{m-1} \mathbf{P}(A_j)(t_{j+1} - t_j) \|Q^{\frac{1}{2}} \Phi_j^*\|_{L_2(H,U)}^2 = \sum_{j=0}^{m-1} \mathbf{P}(A_j)(t_{j+1} - t_j) \|\Phi_j Q^{\frac{1}{2}}\|_{L_2(U,H)}^2 \end{aligned}$$

and thus

$$\mathbf{E}[|I_t^W(\Psi)|_H^2] = \sum_{j=0}^{m-1} (t_{j+1} - t_j) \mathbf{E}[\|\mathbb{1}_{A_j} \Phi_j Q^{\frac{1}{2}}\|_{L_2(U,H)}^2] = \int_0^t \mathbf{E}\|\Psi Q^{\frac{1}{2}}\|_{L_2(U,H)\mathrm{d}s}^2$$

The RHS is finite $\|\Psi Q^{\frac{1}{2}}\|_{L_2(U,H)}^2 \leq \|\Psi\|_{L(U,H)} \|Q^{\frac{1}{2}}\|_{L_2(U)}^2 = \|\Psi\|_{L(U,H)} \mathrm{Tr}(Q)$. □

(Note: This stochastic integral can be defined for any U-valued square-integrable, mean-zero $\mathcal{F}_t$ -Lévy process.)

Next define: $\chi_T := L^2(\Omega \times [0,T], \mathcal{F} \times \mathcal{B}([0,T], \mathrm{d}\mathbf{P} \times \mathrm{d}t; L_2(U_0, H))$ and observe that the RHS of (4.3) is the norm on χ_T when $t = T$.

Then there exists a unique continuous extention of I_t^W, still denoted I_t^W, from the closure of $S(U,H)$ in χ_T into $L^2(\Omega, \mathcal{F}, \mathbf{P}; H)$. This closure has an explicit construction. Before that lets recall the following definition:

Recall: The σ-algebra on $\Omega \times [0,T]$ generated by $A \times (s,t]$ for all $A \in \mathcal{F}_s$, $0 \le s < t \le T$ is called the **predictable** σ-field and is denoted by $P_{[0,T]}$. That is,

$$P_{[0,T]} = \sigma(\{A \times (s,t] : A \in \mathcal{F}_s, 0 \le s < t \le T\}).$$

A function on $\Omega \times [0,T]$ is said to be predictable if it is $P_{[0,T]}$-measurable.

Lemma 4.10. *The closure of $S(U,H)$ in the space χ_T is the subspace of predictable processes in χ_T. In other words, $S(U,H)$ is dense in the space*

$$L^2_{U_0,T}(H) := L^2(\Omega \times [0,T], \mathcal{P}_{[0,T]}, \mathrm{d}\mathbf{P} \times \mathrm{d}t; L_2(U_0,H)),$$

where $\mathcal{P}_{[0,T]}$ is the σ-field of predictable sets.

Remark 4.3. *The space $L^2_{U_0,T}(H)$ of integrands for stochastic integration with respect to W depends (on W because of U_0 and) on the filtration $(\mathcal{F}_t)_{t\ge 0}$ through the requirement of predictability.*

Theorem 4.11. *(PZ Theorem 8.7) Let M be a square-integrable, mean-zero, U-valued $\mathcal{F}_t$ - Lévy process.*

(1) For every $t \in [0,T]$, $I^M_t : L^2_{U_0,t}(H) \to L^2(\Omega; H)$ is an isometry, i.e.,

$$\mathbf{E}(I^M_t(\Psi), I^M_t(\Phi))_H = \mathbf{E}\int_0^t (\Psi(s), \Phi(s))_{L_2(U_0,H)} \mathrm{d}s \tag{4.4}$$

and $\mathbf{E}|I^M_t(\Psi)|^2_H = \mathbf{E}\int_0^t \|\Psi(s)\|^2_{L_2(U_0,H)}\,\mathrm{d}s$ for all $\Psi, \Phi \in \mathbf{L}^2_{U_0,t}(H)$.

(2) For every $\Psi \in L^2_{U_0,T}(H)$ the process $\left(I^M_t(\Psi)\right)_{t\in[0,T]}$ is a square-integrable H-valued martingale that begins at 0. (Rockner Prop 2.3.2: continuous for W)

(3) For every $\Psi \in L^2_{U_0,T}(H)$ the angle bracket of $\left(I^M_t(\Psi)\right)_{t\in[0,T]}$ is given by the formula

$$\left\langle I^M(\Psi)\right\rangle_t = \int_0^t \|\Psi(s)\|^2_{L_2(U_0,H)}\,\mathrm{d}s. \tag{4.5}$$

Thus the BDG inequality for stochastic integrals with respect to W gives: for every $1 \le p < \infty$ there exists a constant $C_p > 0$ such that for every $\mathcal{F}_t$-stopping time τ and every $\Psi \in L^2_{U_0,T}(H)$ we have

$$\mathbf{E}\sup_{t\in[0,\tau]}\Big|\int_0^t \Psi(s)\mathrm{d}W(s)\Big|^p_H \le C_p \mathbf{E}\Big(\int_0^\tau \|\Psi(s)\|^2_{L_2(U_0,H)}\,\mathrm{d}s\Big)^{p/2}. \tag{4.6}$$

(4) Let $A \in L(H,V)$ where V is a real, separable Hilbert space. For every $\Psi \in L^2_{U_0,T}(H)$ we have $A\Psi \in L^2_{U_0,T}(V)$ and $AI^M_t(\Psi) = I^M_t(A\Psi)$. That is, bounded operators can be passed inside the stochastic integral.

Another example is that of a compensated CPP $\widehat{P}$ (it is mean-zero, square-integrable, $\mathcal{F}_t$-adapted) if the CPP P is square integrable.

In the main SPDE (1.1) we consider: $\int_0^t G(u(s-))\mathrm{d}W(s)$. Now does $G(u(s-)) \in L^2_{U_0,T}(H)$? We will take $G : H \to L_2(U_0,H)$ (Borel measurable?) (plus more conditions for existence and uniqueness of solutions to (1.1)). If u is adapted and càdlàg then $u(s-)$ is predictable.

5. Week 5

5.1. **Stochastic integration w.r.t Poisson Random Measure. Recall**: Last time we studied the stochastic integration $I_t^W(\Psi)$ with respect to a Wiener process W (or more generally w.r.t a square integrable $\mathcal{F}_t$-Lévy process M). However using this definition does not help find the jumps of the process $I_t^M(\Psi)$. So we will use the Lévy -Khinchin theorem to decompose M into its purely continuous and purely discontinuous parts (W and a CCPP). So next in our agenda is to define and study stochastic integrals w.r.t a CCPP.

Recall that given a Lévy process L with Lévy measure ν, for each set $A \in \mathcal{B}(U)$ separated from 0, we define $\pi((0,t],A) := \#\{s \in (0,t] : \Delta L(s) \in A\}$. Then π is a Poisson random measure on $[0,\infty) \times U$ with intensity measure $\mathrm{d}t \times \mathrm{d}\nu$. In particular,

$$\pi((s,t],A) = \pi((0,t],A) - \pi((0,s],A) \sim \mathrm{Pois}((t-s)\nu(A)); \tag{5.1}$$

π here is an example of Poisson random measure and the process $\widehat{\pi}((0,t],A) := \pi((0,t],A) - t\nu(A)$ is called a compensated Poisson random measure.

A more general definition is as follows: Let $\mathcal{M}$ be the collection of non-negative, possibly infinite, $\bar{\mathbb{N}}$-valued measures on $(E,\mathcal{E})$, and $\mathcal{B}_{\mathcal{M}}$ be the smallest σ-field on $\mathcal{M}$ such that the maps from $\mathcal{M}$ to $\bar{\mathbb{N}}$ given by $\mu \mapsto \mu(\Gamma)$ are measurable for every $\Gamma \in \mathcal{E}$.

Definition 5.1. *A Poisson random measure on $(E,\mathcal{E})$ with intensity measure λ is an $(\mathcal{M},\mathcal{B}_{\mathcal{M}})$-valued random variable π defined on the probability space $(\Omega,\mathcal{F},\mathbf{P})$ with the property that for all pairwise disjoint sets $\Gamma_1,..,\Gamma_n \in \mathcal{E}$ the $\bar{\mathbb{N}}$-valued random variables $\pi(\Gamma_1),...,\pi(\Gamma_n)$ are independent Poisson random variables with parameters $\lambda(\Gamma_1),...,\lambda(\Gamma_n)$. We call $\widehat{\pi} := \pi - \lambda$ the compensated Poisson random measure.*

Space of integrands:
We consider the following spaces of functions for integration with respect to the Poisson random measure π induced by L:

$$F^p_{\nu,T}(H) = L^p(\Omega \times [0,T] \times E, P_{[0,T]} \times \mathcal{E}, \mathrm{d}\mathbf{P} \times \mathrm{d}t \times \mathrm{d}\nu; H). \tag{5.2}$$

We are only interested in the cases $p = 1,2$; ν is not necessarily finite and so we might not have $F^2_{\nu,T}(H) \subset F^1_{\nu,T}(H)$.
The main fact is that we are able to integrate functions $f \in F^1_{\nu,T}(H)$ with respect to the measure π for $\mathbf{P}$-a.e. fixed $\omega \in \Omega$.

Remark 5.1. *$F^1_{\nu,T}(H) \cap F^2_{\nu,T}(H)$ is dense in both.*

Theorem 5.2. (Part A)

(1) $\forall\, f \in F^1_{\nu,T}(H)$ we have

$$\mathbf{E}\int_{(0,t]}\int_E |f(s,\xi)|_H \mathrm{d}\pi(s,\xi) = \mathbf{E}\int_0^t\int_E |f(s,\xi)|_H \mathrm{d}\nu(\xi)\mathrm{d}s. \quad \forall t \in [0,T] \tag{5.3}$$

(2) $\forall\, f \in F^1_{\nu,T}(H)$ we have for each $t \in [0,T]$

$$\int_{(0,t]}\int_E f(s,\xi)\mathrm{d}\pi(s,\xi) = \sum_{s\in(0,t],\Delta L(s)\neq 0} f(s,\Delta L(s)) \tag{5.4}$$

where the LHS is a.s. a well-defined H-valued integral (Bochner for a fixed $\omega \in \Omega$) and the RHS sum is absolutely convergent.

(3) $\forall\, f \in F^1_{\nu,T}(H)$ and each $t \in [0,T]$ we have

$$\mathbf{E}\int_{(0,t]}\int_E f(s,\xi)\mathrm{d}\pi(s,\xi) = \mathbf{E}\int_0^t\int_E f(s,\xi)\mathrm{d}\nu(\xi)\mathrm{d}s. \tag{5.5}$$

Moving on to part B, for $f \in F^1_{\nu,T}(H) \cap F^2_{\nu,T}(H)$, we define

$$\int_{(0,t]}\int_E f(s,\xi)\mathrm{d}\widehat{\pi}(s,\xi) := \int_{(0,t]}\int_E f(s,\xi)\mathrm{d}\pi(s,\xi) - \int_0^t\int_E f(s,\xi)\mathrm{d}\nu(\xi)\mathrm{d}s. \tag{5.6}$$

Theorem 5.3. (Part B)

(1) Hence $\forall\, f \in F^1_{\nu,T}(H) \cap F^2_{\nu,T}(H)$ and each $t \in [0,T]$ we have

$$\int_{(0,t]}\int_E f(s,\xi)\mathrm{d}\widehat{\pi}(s,\xi) = \sum_{s\in(0,t],\Delta L(s)\neq 0} f(s,\Delta L(s)) - \int_0^t\int_E f(s,\xi)\mathrm{d}\nu(\xi)\mathrm{d}s. \tag{5.7}$$

(2) $\forall\, f \in F^1_{\nu,T}(H) \cap F^2_{\nu,T}(H)$ we have $(\int_{(0,t]}\int_E f(s,\xi)\mathrm{d}\widehat{\pi}(s,\xi))_{t\in[0,T]} \in M^2_T(H)$ and

$$\mathbf{E}|\int_{(0,t]}\int_E f(s,\xi)\mathrm{d}\widehat{\pi}(s,\xi)|^2_H = \mathbf{E}\int_0^t\int_E |f(s,\xi)|^2_H\mathrm{d}\nu(\xi)\mathrm{d}s = \|f\|^2_{F^2_{\nu,t}(H)}. \tag{5.8}$$

(3) Given $f \in F^2_{\nu,T}(H)$, let $(f_n)_{n=1}^\infty$ be a sequence in $F^1_{\nu,T}(H) \cap F^2_{\nu,T}(H)$ that converges to f in $F^2_{\nu,T}(H)$. By Theorem (5.3) (2) the sequence $\left(\int_{(0,t]}\int_E f_n \mathrm{d}\widehat{\pi}\right)_{n=1}^\infty$ is Cauchy in the space of square-integrable H-valued martingales. Furthermore, the limit does not depend on the particular sequence $(f_n)_{n=1}^\infty$. Therefore, we can define the H-valued process $\left(\int_{(0,t]}\int_E f(s,\xi)\mathrm{d}\widehat{\pi}(s,\xi)\right)_{t\in[0,T]}$ to be the limit of any such sequence. By construction, this is a square-integrable H-valued martingale (i.e. $\in M^2_T(H)$) and equation (5.8) continues to hold.

Definition 5.4. *For $T > 0$ we define a map $I^{\widehat{\pi}}_T \colon F^2_{\nu,T}(H) \to L^2(\Omega; H)$ by*

$$I^{\widehat{\pi}}_T(f) := \int_{(0,T]}\int_E f(s,\xi)\mathrm{d}\widehat{\pi}(s,\xi). \tag{5.9}$$

Equation (5.8) says that $I^{\widehat{\pi}}_T$ is an isometry.

Remark 5.2. *The stochastic integration with respect to the Poisson random measure π is only defined for integrands in $F^1_{\nu,T}(H)$, while the stochastic integration with respect to the compensated Poisson random measure $\widehat{\pi}$ is only defined for integrands in $F^2_{\nu,T}(H)$. And so the formula*

$$\int_{(0,t]}\int_E f(s,\xi)\mathrm{d}\widehat{\pi}(s,\xi) = \sum_{s\in(0,t],\Delta L(s)\neq 0} f(s,\Delta L(s)) - \int_0^t\int_E f(s,\xi)\mathrm{d}\nu(\xi)\mathrm{d}s, \tag{5.10}$$

is valid only for $f \in F^1_{\nu,T}(H) \cap F^2_{\nu,T}(H)$ (As described earlier the LHS is defined as a limit and so the formula may not hold for $f \in F^2_{\nu,T}(H)$). But if L is compound Poisson process then we know that ν is finite and thus $F^2_{\nu,T}(H) \subset F^1_{\nu,T}(H)$. And thus the formula holds for any $f \in F^2_{\nu,T}(H)$.

Fact: For every $f \in F^2_{\nu,T}(H)$, the stochastic process $(I^{\widehat{\pi}}_t(f))_{t\in[0,T]}$ is a purely discontinuous martingale that starts from 0.

Proof. Recall the definitions (from Week 2): Let $M \in M_T^2(H)$. Then $M_T^2(H) = M_T^{2,c}(H) \oplus M_T^{2,d}(H)$ i.e. M can be uniquely written as

$$M = M^c + M^d$$

where $M^c \in M_T^{2,c}(H)$, is called the purely continuous part of M and $M^d \in M_T^{2,d}(H)$ is called the purely discontinuous part of M. $M_T^{2,c}(H)$ is the subspace of continuous martingales in $M_T^2(H)$ (continuous $\mathbf{P}$-a.s) and $M_T^{2,d}(H)$, is the close of the subspace of bounded variation martingales in $M_T^2(H)$ starting from 0.

Consider $f_n \in F^1_{\nu,T}(H) \cap F^2_{\nu,T}(H)$ such that $f_n \to f$ in $F^2_{\nu,T}(H)$. So for f_n's we have (5.7) hold true i.e.

$$I_t^{\widehat{\pi}}(f_n) = \sum_{s\in(0,t],\Delta L(s)\neq 0} f_n(s, \Delta L(s)) - \int_0^t \int_E f_n(s,\xi)\mathrm{d}\nu(x)\mathrm{d}s$$

where the RHS integral is absolutely continuous. Thus $I_t^{\widehat{\pi}}(f_n)$ has bounded variation and starts at 0. For the f_n's we have the Itô isometry: $\mathbf{E}|I_t^{\widehat{\pi}}(f_n)|_H^2 = \mathbf{E}\int_0^t\int_E |f_n(s,\xi)|_H^2 \mathrm{d}\nu(\xi)\mathrm{d}s$ which implies that $I_t^{\widehat{\pi}}(f_n) \to I_t^{\widehat{\pi}}(f) \quad$ in $M_T^2(H)$. The desired result follows from the fact that $M^{2,d}$ is closed. □

Now we will present the quadratic variation and BDG inequality for stochastic integrals with respect to the compensated Poisson random measure π.

Theorem 5.5. *(PZ 8.23) Let P be an $\mathcal{F}_t$-compound Poisson point process on a measurable space $(E, \mathcal{E})$ with intensity measure ν and let π denote the associated Poisson random measure. For every $f \in F^2_{\nu,T}(H)$ the quadratic variation of the H-valued martingale $\left(I_t^{\widehat{\pi}}(f)\right)_{t\in[0,T]}$ is given by*

$$[I^{\widehat{\pi}}(f)]_t = \int_{(0,t]}\int_E |f(s,\xi)|_H^2 \mathrm{d}\pi(s,\xi). \tag{5.11}$$

Proof. Recall the relation between the angle and square brackets: $[M]_t = < M^c >_t + \sum_{s<t}|\Delta M(s)|^2$. We have $I_t^{\widehat{\pi}}(f) = (I_t^{\widehat{\pi}}(f))^d$. And so the proof goes in the following direction:

$$[I_t^{\widehat{\pi}}(f)]_t = \sum_{s<t}|\Delta I_s^{\widehat{\pi}}(f)|^2 = \sum_{s\in(0,t]}|f(s,\Delta L(s))|_H^2 = \int_0^t\int_E |f|_H^2 \mathrm{d}\pi(\xi,s).$$

□

Now the BDG inequality:

Proposition 5.6. *In the setup above, for every $1 \le p < \infty$ there exists a constant $C_p \in (0,\infty)$ such that for every $\mathcal{F}_t$-stopping time τ and for every $f \in \mathbf{F}^2_{\nu,T}(H)$ we have*

$$\mathbf{E}\sup_{t\in[0,\tau]}\Big|\int_{(0,t]}\int_E f(s,\xi)\mathrm{d}\widehat{\pi}(s,\xi)\Big|_H^p \le C_p\mathbf{E}\Big(\int_{(0,\tau]}\int_E |f(s,\xi)|_H^2\mathrm{d}\pi(s,\xi)\Big)^{p/2}. \tag{5.12}$$

C_p is independent of τ or f.

5.2. **Relation with integration w.r.t a Lévy process.** So far we have seen how to integrate w.r.t a square-integrable, mean-zero $\mathcal{F}_t$-Lévy process. We also studied how to integrate w.r.t a compensated Poisson random measure.

Question: How do these concepts relate to integration w.r.t a Lévy process?

Let's revisit the Lévy -Khinchin formula for a Lévy process L with Lévy measure ν: We have $\int_U (|y|_U^2 \wedge 1) \mathrm{d}\nu(y) < \infty$ and

$$L(t) = at + \underbrace{W(t) + \sum_{n=1}^{\infty} \widehat{L}_{A_n}(t)}_{\text{square-integrable Lévy martingale}} + L_{A_0}(t) \tag{5.13}$$

where W is a Wiener process, and the L_{A_n}'s are CPPs (all independent of each other). (Here the $(L_{A_n})_1^\infty$ are square-integrable (not L_{A_0}). This follows from Lemma 4.2 part (1).)

Here are some consequences:

- Let M_0 and M_1 be two square integrable, mean-zero $\mathcal{F}_t$-Lévy processes. Then $M = M_0 + M_1$ has the covariance operator $Q = Q_0 + Q_1$ and for $\Psi \in L^2_{Q^{\frac{1}{2}}(U),T}(H)$

$$\int_0^t \Psi(s) \mathrm{d}M(s) = \int_0^t I_0(\Psi)(s) \mathrm{d}M_0(s) + \int_0^t I_1(\Psi)(s) \mathrm{d}M_1(s) \tag{5.14}$$

  where $I_j : L^2_{Q^{\frac{1}{2}}(U),T}(H) \to L^2_{Q_j^{\frac{1}{2}}(U),T}(H)$ $j = 0, 1$ is a continuous extension of the identity map (from and to $S(U, H)$). Using this, for a square-integrable process M, we can write I_t^M in terms of its continuous part I_t^W and discontinuous part $I_t^{\mathscr{P}}$ ($\mathscr{P}$ being the sum of CCPP's) and we can thus explicitly find the quadratic variations.
- Now lets turn our focus to the term $I_t^{\mathscr{P}}$. So far, we have seen: Integral w.r.t a square-integrable process, the isometry $I_T^{\mathscr{P}} : L^2_{Q_1^{\frac{1}{2}}(U),T}(H) \to L^2(\Omega, H)$. And integral w.r.t a compensated PRM, the isometry $I_T^{\widehat{\pi}}(H) : F^2_{\nu,T}(H) \to L^2(\Omega, H)$. Then we have the informal decomposition $\mathrm{d}\mathscr{P} := \mathrm{d} \sum_1^\infty \widehat{L}_{A_n} = \mathrm{d}\widehat{\pi}$ (π is the jump measure of $\mathscr{P}$). The proof for this goes as follows:
- Assume that Q_1 is the covariance operator and π is the jump measure of $\mathscr{P}$. There is an isometry

$$f^{\mathscr{P}} : L^2_{Q_1^{\frac{1}{2}}(U),T}(H) \to F^2_{\nu,T}(H).$$

  In the proof, for a simple function Ψ we can show that $f^{\mathscr{P}}(\Psi) = f_\Psi^{\mathscr{P}} \in F^2_{\nu,T}(H)$ where $f_\Psi^{\mathscr{P}} : \Omega \times [0, \infty) \times U \to H$ is defined as $f_\Psi^{\mathscr{P}}(\omega, s, u) = \Psi(s, \omega)(u))$. By a density argument we get for any $\Psi \in L^2_{Q_1^{\frac{1}{2}}(U),T}(H)$ that $f_\Psi^{\mathscr{P}} \in F^2_{\nu,T}(H)$.
- Then we restrict the analysis to the case when $\mathscr{P}$ is a CCPP $\widehat{P}$ (as opposed to a sum of CCPP's) and show that $I_t^{\widehat{P}} = I_t^{\widehat{\pi}} \circ f^{\widehat{P}}$ where π is the jump measure of P. That is, for every $t \in [0, T]$,

$$\int_0^t \Psi(s) \mathrm{d}\widehat{P}(s) = \int_0^t \int_U f_\Psi^{\widehat{P}}(s, u) \mathrm{d}\widehat{\pi}(s, u) \tag{5.15}$$

$$= \sum_{s \in (0,T]} f_\Psi^{\widehat{P}}(s, \Delta P(s)) - \int_0^t \int_U f_\Psi^{\widehat{P}}(s, u) \mathrm{d}\nu(u) \mathrm{d}s. \tag{5.16}$$

- Then by a limiting argument we extend the above equation to the general case i.e. for $\Psi \in L^2_{Q_1^{\frac{1}{2}}(U),T}(H)$ we have $I_t^{\mathscr{P}}(\Psi) = I_t^{\widehat{\pi}}(f_\Psi^{\mathscr{P}})$. Note that (5.16) may not hold in this case.

- Hence, for a square-integrable Lévy process L we have

$$I_T^L(\Psi) = I_T^W(I_0(\Psi)) + I_T^{\widehat{\pi}}(f^{\mathscr{P}}_{I_1(\Psi)})$$

  and in fact,

$$\Delta I_t^L(\Psi) = f^{\mathscr{P}}_{I_1(\Psi)}(t, \Delta L(t)).$$

Thus, now that we have identified jumps of a square-integrable $\mathcal{F}_t$-Lévy process we get an explicit Itô's formula. But before that we note the following:

Remark 5.3. *L may not be square-integrable in which case we need to figure out a way to define integral w.r.t* $\mathrm{d}P_0$ *where $P_0 = L_{A_0}$ is the $\mathcal{F}_t-$CPP in the Lévy -Khintchin decomposition.*

Summary: It is done as follows: Up until the time that the jumps of P_0 leave the ball $B(0, m) \subset U$, denoted τ_m, P_0 agrees with the square-integrable compound Poisson process P_m formed by taking from P_0 only the jumps that lie in $B(0, m)$. Define: $\int_0^t \Psi(s)\mathrm{d}P_m(s) = \sum_{s\in(0,t]} f_\Psi^{\widehat{P}_m}(s, \Delta P_m(s))$ for Ψ in an appropriate space such that all these integrals are defined. Then define $\int_0^t \Psi(s)\mathrm{d}P(s)$ piecewise to be equal to $\int_0^t \Psi(s)\mathrm{d}P_m(s)$ on the interval $\{t < \tau_m\}$ (for appropriate Ψ). It can then be shown that for every Ψ, there exists a $\mathcal{P}_{[0,T]} \otimes \mathcal{B}(U)$-measurable function $f_\Psi^{P_0} : \Omega \times [0, T] \times U \to H$ such that

$$\int_0^t \Psi(s)\mathrm{d}P_0(s) = \int_{(0,t]} \int_U f_\Psi^{P_0}(s, u)\mathrm{d}\pi(s, u) = \sum_{s\in(0,t]} f_\Psi^{P_0}(s, \Delta P_0(s)) \tag{5.17}$$

And thus for $\Psi \in L^2(\Omega \times [0, T], \mathcal{P}_{[0,T]}, \mathrm{d}\mathbf{P} \otimes \mathrm{d}t; L(U, H))$ (the appropriate space) we finally obtain

$$\int_0^t \Psi(s)\mathrm{d}L(s) = \int_0^t \Psi a \mathrm{d}s + \int_0^t \Psi(s)\mathrm{d}W(s) + \int_{(0,t]} \int_{B_1} f_\Psi^{\mathscr{P}}(s, u)\mathrm{d}\widehat{\pi}(s, u) + \int_{(0,t]} \int_{B_1^c} f_\Psi^{P_0}(s, u)\mathrm{d}\pi(s, u). \tag{5.18}$$

And so in our SPDE driven by Lévy noise we consider integrals of the kind,

$$\int_0^t G(u(s-))\mathrm{d}W(s) + \int_{(0,t]} \int_{E_0} K(u(s-), \xi)\mathrm{d}\widehat{\pi}(s, \xi) + \int_{(0,t]} \int_{E\setminus E_0} \mathcal{K}(u(s-), \xi)\mathrm{d}\pi(s, \xi).$$

6. Week 6

Recap:So now on we consider will consider processes/SPDE of the following form:

$$X(t) = X_0 + \int_0^t F(s)\mathrm{d}s + \int_0^t G(s)\mathrm{d}W(s) + \int_{(0,t]}\int_E K(s,\xi)\mathrm{d}\widehat{\pi}(s,\xi) + \int_{(0,t]}\int_{E\setminus E_0} \mathcal{K}(s,\xi)\mathrm{d}\pi(s,\xi).$$

As an example, we can set $E := U \setminus \{0\}$, $E_0 := \{y \in U : 0 < |y|_U < 1\}$ and endow the space E with a metric such that it is separable, complete, such that $A \subset E$ is bounded if and only if A is separated from 0, and ν is finite on bounded subsets of E.
In today's class we will state the Itô formula for Hilbert space valued processes and prove the 1-d version of it.
We first consider the processes of the form:

$$X(t) = X(0) + \int_0^t \int_A \mathcal{K}(s,\xi)\mathrm{d}\pi(s,\xi).$$

$\mathcal{K}$ is predictable and A is set in E bounded below.

Theorem 6.1. *Then for each $f \in C(\mathbb{R})$ and for each $t \geq 0$ with probability 1 we have:*

$$f(X(t)) - f(X(0)) = \int_0^t \int_A [f\Big(X(s-) + \mathcal{K}(s,\xi)\Big) + f(X(s-))]\mathrm{d}\pi(s,\xi).$$

Proof. We have seen previously that $P(t) = \sum_s \Delta L(s)\chi_A(\Delta L(s))$ is a CPP with Lévy measure $\nu|_A$. We define a partition formed by the jumps of P i.e. we set $T_n = \inf\{t > T_{n-1} : \Delta P(t) \in A\}$.

$$\begin{aligned} f(X(t)) - f(X(0)) &= \sum_{0\leq s\leq t} [f(X(s)) - f(X(s-))] \\ &= \sum_{n=1}^{\infty} [f(X(t\wedge T_n)) - f(X(t\wedge T_{n-1}))] \\ &= \sum_{n=1}^{\infty} [f(X(t\wedge T_n-) + \mathcal{K}(t\wedge T_n, \Delta P(t\wedge T_n))) - f(X(t\wedge T_n-))] \\ &= \int_0^t \int_A [f\left(X(s-) + \mathcal{K}(s,\xi)\right) - f(X(s-))]\mathrm{d}\pi(s,\xi). \end{aligned}$$

□

Now, consider the process M of the form

$$M(t) = \int_0^t F(s)\mathrm{d}s + \int_0^t G(s)\mathrm{d}B \tag{6.1}$$

where B is a 1-D Brownian motion. Then the quadratic variation of M is $[M](t) = \int_0^t G^2(s)\mathrm{d}s$.

Theorem 6.2. *For M as in (6.1) we have for any $f \in C^2(\mathbb{R})$:*

$$f(M(t)) - f(M(0)) = \int_0^t f'(M(s))\mathrm{d}M(s) + \frac{1}{2}\int_0^t f''(M(s))\mathrm{d}[M](s). \tag{6.2}$$

Proof. (Outline) (First assuming that M, F, G are bounded r.v.s) Let $(P_n, n \in \mathbb{N})$ be a sequence of partitions of the form $P_n = \{0 = t_0^n < ... < t_{m+1}^n = t\}$ (m depends on n) and suppose that $\lim_{n\to\infty} \delta(P_n) = 0$, where the mesh $\delta(P_n)$ is given by $\max_{0\leq j\leq m}|t_{j+1}^n - t_j^n|$. By Taylor's theorem we have, for each such partition

$$f(M(t)) - f(M(0)) = \sum_{k=0}^{m} f(M(t_{k+1})) - f(M(t_k)) =: J_1(t) + \frac{1}{2} J_2(t)$$

where

$$J_1(t) = \sum_{k=0}^{m} f'(M(t_k))(M(t_{k+1}) - M(t_k)), \quad J_2(t) = \sum_{k=0}^{m} f''(N^k)(M(t_{k+1}) - M(t_k))^2$$

where N^k's are $\mathcal{F}_{t_{k+1}}$-adapted random variables satisfying $|N^k - M(t_k)| \leq |M(t_{k+1}) - M(t_k)|$. Then we can show that as $n \to \infty$

$$J_1(t) \to \int_0^t f'(M(s)) \mathrm{d}M(s) \quad \text{in probability.}$$

For J_2 we write $\quad J_2(t) = K_1(t) + K_2(t)$

$$=: \sum_{k=0}^{m} f''(M(t_k))[M(t_{k+1}) - M(t_k)]^2 + \sum_{k=0}^{m} (f''(N^k) - f''(M(t_k)))[M(t_{k+1}) - M(t_k)]^2.$$

Then we can show that

$$K_1 \to \int_0^t f''(M(s)) \mathrm{d}[M](s).$$

To treat K_2 observe:

$$|K_2(t)| \leq \max_{0 \leq k \leq m} |f''(N^k) - f''(M(t_k))| \cdot \sum_{k=0}^{m} (M(t_{k+1}) - M(t_k))^2.$$

By continuity $\max_{0 \leq k \leq m} |f''(N^k) - f''(M(t_k))| \to 0$ while $\sum_{k=0}^{m} (M(t_{k+1}) - M(t_k))^2 \to [M](t)$ in L^2. And thus $K_2 \to 0$ in probability. □

Next, we consider the form

$$X(t) = X(0) + \underbrace{\int_0^t F(s) \mathrm{d}s + \int_0^t G(s) \mathrm{d}B(s)}_{X_c(t)} + \underbrace{\int_0^t \int_A \mathcal{K}(s, \xi) \mathrm{d}\pi(s, \xi)}_{X_d(t)},$$

where B is 1-D Brownian motion.

Theorem 6.3. *Then, for all $f \in C^2(\mathbb{R})$, and $t \geq 0$ a.s.*

$$\begin{aligned} f(X(t)) - f(X(0)) &= \int_0^t f'(X(s-)) \mathrm{d}X_c(s) + \frac{1}{2} \int_0^t f''(X(s-)) \mathrm{d}[X_c](s) \\ &\quad + \int_0^t \int_A [f(X(s-) + \mathcal{K}(s, \xi)) - f(X(s-))] \mathrm{d}\pi(s, \xi) \end{aligned}$$

Proof. For T_j's as defined in the previous theorem we have

$$\begin{aligned} f(X(t)) - f(X(0)) &= \sum_{j=0}^{\infty} [f(X(t \wedge T_{j+1})) - f(X(t \wedge T_j))] \\ &= \sum_{j=0}^{\infty} [f(X(t \wedge T_{j+1})) - f(X(t \wedge T_{j+1}-))] + [f(X(t \wedge T_{j+1}-)) - f(X(t \wedge T_j))] \end{aligned}$$

The first sum is equal to $\int_0^t \int_A [f(X(s-)+\mathcal{K}(s,\xi)) - f(X(s-))]\mathrm{d}\pi(s,\xi)$ from Theorem 6.1. For the second sum consider $s \in [T_j, T_{j+1}]$. Then,

$$X(s-) - X(T_j) = X_c(s) - X_c(T_j)$$

Now we can apply Theorem 6.2 to the process $X(s-)$:

$$f(X(s-)) - f(X(T_j)) = \int_{T_j}^{s-} f'(X(r-))\mathrm{d}X_c(r) + \frac{1}{2}\int_{T_j}^{s-} f''(X(r-))\mathrm{d}[X_c](r).$$

Then take $s = t \wedge T_{j+1}$ and $\sum_{j=0}^{\infty}$ to get that the second sum is equal to $\int_0^t f'(X(s-))\mathrm{d}X_c(s) + \frac{1}{2}\int_0^t f''(X(s-))\mathrm{d}[X_c](s)$. Thus we have the desired result.
Note that in the 1D case this result can also be stated as:

$$\begin{aligned} f(X(t)) - f(X(0)) = \int_0^t f'(X(s-))F(s)\mathrm{d}s + \int_0^t f'(X(s-))G(s)\mathrm{d}B(s) + \frac{1}{2}\int_0^t f''(X(s-))G^2(s)\mathrm{d}s \\ + \int_0^t \int_A [f(X(s-)+\mathcal{K}(s,\xi)) - f(X(s-))]\mathrm{d}\pi(s,\xi). \end{aligned} \tag{6.3}$$

□

Now consider the general Lévy -type stochastic integral form:

$$X(t) = X(0) + \underbrace{\int_0^t F(s)\mathrm{d}s + \int_0^t G(s)\mathrm{d}B(s)}_{:=X_c} + \int_{|\xi|<1} K(s,\xi)\mathrm{d}\widehat{\pi}(s,\xi) + \int_{|\xi|\geq 1} \mathcal{K}(s,\xi)\mathrm{d}\pi(s,\xi). \tag{6.4}$$

Before stating a more general Itô formula we state the following lemma:

Lemma 6.4. *Let $K \in F^2_{\nu,T}$ then for any sequence $(A_n)_{n\in N} \uparrow E$ we have*

$$\lim_{n\to\infty} \int_{(0,t]}\int_{A_n} K(s,\xi)\mathrm{d}\widehat{\pi}(s,\xi) = \int_{(0,t]}\int_E K(s,\xi)\mathrm{d}\widehat{\pi}(s,\xi) \text{ in probability.}$$

Corollary: Let

$$X_n(t) = X(0) + \int_0^t F(s)\mathrm{d}s + \int_0^t G(s)\mathrm{d}B(s) + \int_{(0,t]}\int_{A_n} K(s,\xi)\mathrm{d}\widehat{\pi}(s,\xi) + \int_{(0,t]}\int_{|\xi|\geq 1} \mathcal{K}(s,\xi)\mathrm{d}\pi(s,\xi)$$

Then $X_n \to X$ in probability.

Theorem 6.5. *Then, for all $f \in C^2(\mathbb{R})$ and $t \geq 0$, a.s.*

$$\begin{aligned} f(X(t)) - f(X(0)) &= \int_0^t f'(X(s-))\mathrm{d}X_c(s) + \frac{1}{2}\int_0^t f''(X(s-))\mathrm{d}[X_c](s) \\ &+ \int_{(0,t]}\int_{|\xi|\geq 1} [f\Big(X(s-)+\mathcal{K}(s,\xi)\Big) - f(X(s-))]\mathrm{d}\pi(s,\xi) \\ &+ \int_0^t \int_{|\xi|<1} [f\Big(X(s-)+K(s,\xi)\Big) - f(X(s-)) - K(s,\xi)f'(X(s-))]\mathrm{d}\nu(\xi)\mathrm{d}(s). \end{aligned}$$

Proof. Decompose $|x|< 1$ as the union of sets $\{A_n\}_{n\in\mathbb{N}}$ as in Lemma 6.4 that are bounded from below.
Recall that we have

$$\int_{(0,t]}\int_{A_n} K(s,\xi)\mathrm{d}\widehat{\pi}(s,\xi) = \int_{(0,t]}\int_{A_n} K(s,\xi)\mathrm{d}\pi(s,\xi) - \int_0^t \int_{A_n} K(s,\xi)\mathrm{d}\nu(\xi)\mathrm{d}s.$$

And so applying Theorem 6.5 to X_n defined as:

$$X_n(t) = X_n(0) + \int_0^t F(s)\mathrm{d}s + \int_0^t G(s)\mathrm{d}B(s) + \int_{(0,t]}\int_{A_n} K(s,\xi)\mathrm{d}\pi(s,\xi) - \int_0^t \int_{A_n} K(s,\xi)\mathrm{d}\nu(\xi)\mathrm{d}s + \int_{(0,t]}\int_{|\xi|\geq 1} \mathcal{K}(s,\xi)\mathrm{d}\pi(s,\xi)$$

we obtain

$$\begin{aligned} f(X_n(t)) - f(X_n(0)) &= \int_0^t f'(X(s-))\mathrm{d}X_c(s) + \frac{1}{2}\int_0^t f''(X(s-))\mathrm{d}[X_c](s) \\ &+ \int_{(0,t]}\int_{|x|\geq 1} [f\Big(X(s-) + \mathcal{K}(s,\xi)\Big) - f(X(s-))]\mathrm{d}\pi(s,\xi) \\ &+ \int_{(0,t]}\int_{A_n} [f\Big(X(s-) + K(s,\xi)\Big) - f(X(s-))]\mathrm{d}\pi(s,\xi) - \int_0^t \int_{A_n} K(s,\xi) f'(X(s-))\mathrm{d}\nu(\xi)\mathrm{d}s. \end{aligned}$$

The last two terms can be combined to obtain:

$$\int_0^t \int_{A_n} [f\Big(X(s-) + K(s,\xi)\Big) - f(X(s-))]\mathrm{d}\widehat{\pi}(s,\xi)$$

missing

$$+ \int_0^t \int_{A_n} [f\Big(X(s-) + K(s,\xi)\Big) - f(X(s-)) - K(s,\xi)f'(X(s-))]\mathrm{d}\nu(\xi)\mathrm{d}s.$$

From the abovementioned Corollary we know that $X_n(t) \to X(t)$ in probability and thus there is a subsequence for which $X_n(t) \to X(t)$ a.s. And so we pass to the limit $n \to \infty$ on both sides to get the desired result. □

We will now state the Itô formula for values in a Hilbert space H. Consider an H-valued process of the form:

$$X(t) = X_0 + \int_0^t F(s)\mathrm{d}s + \int_0^t G(s)\mathrm{d}W(s) + \int_{(0,t]}\int_E K(s,\xi)\mathrm{d}\widehat{\pi}(s,\xi), \tag{6.5}$$

where $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbf{P}; H)$, $F\colon \Omega \times [0,T] \to H$ satisfies $F \in L^1(0,T;H)$ a.s., $G \in L^2_{U_0,T}(H)$ and $K \in F^2_{\nu,T}(H)$ for some $T > 0$.

Theorem 6.6. *Let X be as in (6.5). Let $\psi\colon H \to \mathbb{R}$ be a C^2 function such that ψ', ψ'' are uniformly continuous on bounded subsets of H (as a mapping into the space of Hilbert-Schmidt operators on H.) Then for each $t \geq 0$ we have $\mathbf{P}$-a.s.*

$$\begin{aligned} \psi(X(t)) = \ & \psi(X_0) + \int_0^t (\psi'(X(s-)), F(s))_H \mathrm{d}s + \int_0^t (\psi'(X(s-)), G(s)\mathrm{d}W(s))_H \\ &+ \int_{(0,t]}\int_E (\psi'(X(s-)), K(s,\xi))_H \mathrm{d}\widehat{\pi}(s,\xi) + \frac{1}{2}\int_0^t \mathrm{Tr}[\psi''(X(s-))G(s)G^*(s)]\mathrm{d}s \\ &+ \int_{(0,t]}\int_E [\psi(X(s-) + K(s,\xi)) - \psi(X(s-))]\,\mathrm{d}\pi(s,\xi) - \int_{(0,t]}\int_E \langle \psi'(X(s-)), K(s,\xi)\rangle_H \mathrm{d}\pi(s,\xi). \end{aligned}$$

Special case: $\psi(u) := |u|_H^2$. Despite the fact that the second-order derivative of this map is not a Hilbert-Schmidt operator when H is infinite-dimensional, the terms on the right-hand side of the above formula are still well-defined. By projecting X onto finite dimensional subspaces of H, applying Theorem 6.6 and passing to the limit, one can show that above equation equation still holds for $\psi(u) := |u|_H^2$.

Corollary 1. *Let X be as in (6.5). Then for all $t \geq 0$ we have $\mathbf{P}$-a.s.*

$$|X(t)|_H^2 = |X_0|_H^2 + 2\int_0^t (X(s-), F(s))_H \mathrm{d}s + 2\int_0^t (X(s-), G(s)\mathrm{d}W(s))_H$$

$$+ 2\int_{(0,t]}\int_E (X(s-), K(s,\xi))_H \mathrm{d}\widehat{\pi}(s,\xi) + \int_0^t \|G(s)\|^2_{L_2(U_0,H)}\,\mathrm{d}s + \int_{(0,t]}\int_E |K(s,\xi)|_H^2 \mathrm{d}\pi(s,\xi).$$

7. Week 7

7.1. Existence and uniqueness results.

H= separable Hilbert space. We first consider an H valued SDE:

$$\begin{cases} \mathrm{d}Y = F(s, Y(s))\mathrm{d}s + +G(s, Y(s-))\mathrm{d}W(s) + \int_{E_0} K(s, Y(s-), \xi)\mathrm{d}\widehat{\pi}(s,\xi) \\ Y(0) = Y_0. \end{cases} \tag{7.1}$$

The integral w.r.t $\mathrm{d}\pi$ is accounted for using a standard method known as the "piecing method" and so first we will only consider equations of the form (7.1).

Setting:

Stochastic basis: $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P}, W, \pi)$

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ satisfies the usual conditions: $(\mathcal{F}_t)_{t\geq 0}$ is a right-continuous and complete filtration.
- W is a $\mathcal{F}_t$-Wiener process.
- π is a Poisson random measure (induced by an $\mathcal{F}_t$-Poisson point process) on $(0,\infty) \times E$ with intensity measure $\mathrm{d}t \otimes \mathrm{d}\nu$ where ν is a σ-finite measure on $(E, \mathcal{E})$.
- Q = covariance operator of W, $U_0 = Q^{\frac{1}{2}}(U)$.
- E_0 any measurable subset of E.
- $Y_0 : \Omega \to H$ is $\mathcal{F}_0$ measurable r.v.
- $F : \Omega \times [0,T] \times H \to \mathbb{R}$.
- $G : \Omega \times [0,T] \times H \to L_2(U_0, H)$.
- $K : \Omega \times [0,T] \times H \times E \to H$.

Definition 7.1. *For $Y_0 \in L^2(\Omega, \mathcal{F}_0, \mathbf{P}; H)$, an H valued càdlàg adaptive process $(Y(t))_{t\geq 0}$ is said to be a global pathwise solution to (7.1) on $[0,T]$ if*

(1) $Y \in L^2(\Omega; L^2(0,T;H))$

(2) $\forall t \in [0,T]$ *the following holds* $\mathbf{P}$*-a.s.*

$$Y(t) = Y_0 + \int_0^t F(s, Y(s))\mathrm{d}s + \int_0^t G(s, Y(s-))\mathrm{d}W(s) + \int_{(0,t]}\int_{E_0} K(s, Y(s-), \xi)\mathrm{d}\widehat{\pi}(s,\xi). \tag{7.2}$$

Definition 7.2. *Let τ be a stopping time. Then the pair (Y, τ) is said to be a local pathwise solution to (7.1) up to τ if (7.2) holds on the interval $\{t < \tau\}$.*

7.2. **Global existence for SDE with globally Lipschitz and linear growth conditions.** Assumptions: For some $\rho > 0$:

$$|F(s,x)|_H^2 + \|G(s,x)\|^2_{L_2(U_0,H)} + \int_{E_0} |K(s,x,\xi)|_H^2 \mathrm{d}\nu(\xi) \le \rho(1+|x|_H^2) \tag{7.3}$$

$$|F(s,x)-F(s,y)|_H^2 + \|G(s,x)-G(s,y)\|^2_{L_2(U_0,H)} + \int_{E_0} |K(s,x,\xi)-K(s,y,\xi)|_H^2 \mathrm{d}\nu(\xi) \le \rho(|x-y|_H^2) \tag{7.4}$$

$\forall x \in H, s \ge 0, \omega \in \Omega$.

Lemma 7.3. *Under the above assumptions the integrals* $\int_0^t F(s,Y(s))\mathrm{d}s$, $\int_0^t G(s,Y(s-))\mathrm{d}W(s)$ *and,* $\int_{(0,t]}\int_{E_0} K(s,Y(s-),\xi)\mathrm{d}\widehat{\pi}(s,\xi)$ *are well-defined if* $Y \in L^2(\Omega\times[0,T], \mathcal{F}\times\mathcal{B}([0,T]), \mathrm{d}\mathbf{P}\otimes\mathrm{d}t; H)$.

Indeed thanks to Itô isometry we have

$$\begin{aligned}
\mathbf{E}|\int_0^t F(s,Y(s))\mathrm{d}s|_H^2 &\le \int_0^t \mathbf{E}(1+|Y(s)|_H^2) < \infty \\
\mathbf{E}|\int_0^t G(s,Y(s-))\mathrm{d}W(s)|_H^2 &= \mathbf{E}\int_0^t \|G(s,Y(s))\|^2_{L_2(U_0,H)} \\
&\le \int_0^t \mathbf{E}(1+|Y(s-)|_H^2) < \infty \\
\mathbf{E}(|\int_{(0,t]}\int_{E_0} K(s,Y(s-),\xi)\mathrm{d}\widehat{\pi}(s,\xi)|_H^2) &= \mathbf{E}\int_{(0,t]}\int_{E_0} |K(s,Y(s-),\xi)|_H^2 \mathrm{d}\nu(\xi)\mathrm{d}s \\
&\le \int_0^t \mathbf{E}(1+|Y(s-)|_H^2) < \infty
\end{aligned}$$

Note: In fact if Y is $\mathcal{F}_t$-adapted and càdlàg then so are the integrals. The fact that they are mean-zero martingales (as established in earlier classes) and that the filtration satisfies the usual conditions imply that they have a càdlàg modification.

Recall BDG inequalities:

1. **For W:** $1 \le p < \infty$ there exists a constant $C_p > 0$ such that for every $\Psi \in L^2_{U_0,T}(H)$ we have

$$\mathbf{E}\sup_{t\in[0,\tau]}\Big|\int_0^t \Psi(s)\mathrm{d}W(s)\Big|_H^p \le C_p\mathbf{E}\Big(\int_0^\tau \|\Psi(s)\|^2_{L_2(U_0,H)}\,\mathrm{d}s\Big)^{p/2}. \tag{7.5}$$

2. **For $\widehat{\pi}$:** For every $f \in \mathbf{F}^2_{\nu,T}(H)$ we have

$$\mathbf{E}\sup_{t\in[0,\tau]}\Big|\int_{(0,t]}\int_E f(s,\xi)\mathrm{d}\widehat{\pi}(s,\xi)\Big|_H^p \le C_p\mathbf{E}\Big(\int_{(0,\tau]}\int_E |f(s,\xi)|_H^2\mathrm{d}\pi(s,\xi)\Big)^{p/2}. \tag{7.6}$$

Theorem 7.4. *(Ikeda- Watanabe Chapter 4 Theorem 9.1/Applebaum 6.2.3) In the setup above, for every* $Y_0 \in L^2(\Omega,\mathcal{F}_0,\mathbf{P};H)$ *and every* $T>0$ *there exists a global solution* Y *to the SDE*

$$\begin{cases} \mathrm{d}Y = F(s,Y(s-))\mathrm{d}s + G(s,Y(s-))\mathrm{d}W(s) + \int_{E_0} K(s,Y(s-),\xi)\mathrm{d}\widehat{\pi}(s,\xi), \\ Y(0) = Y_0. \end{cases} \tag{7.7}$$

on the interval $[0,T]$ *with initial condition* Y_0.

Proof. We use Picard iteration. Start with $Y_0(t) = Y_0$, then iterate, for $n \geq 1$, define

$$Y_n(t) = Y_0 + \int_0^t F(s, Y_{n-1}(s-))\mathrm{d}s + \int_0^t G(s, Y_{n-1}(s-))\mathrm{d}W(s) + \int_{(0,t]}\int_{E_0} K(s, Y_{n-1}(s-), \xi)\mathrm{d}\widehat{\pi}(s,\xi). \tag{7.8}$$

Y_n is well-defined, adapted and càdlàg provided Y_{n-1} is adapted and càdlàg .
Aim: Show that the sequence $(Y_n)_1^\infty$ converges in $L^2(\Omega; L^\infty([0,T]; H))$. Hence we will show that it's Cauchy. **Estimates:** $Y_{n+1}(t) - Y_n(t) = I_1 + I_2 + I_3$.

$$I_1 = \int_0^t (F(Y_n(s)) - F(Y_{n-1}(s)))\mathrm{d}s$$

$$I_2 = \int_0^t (G(Y_n(s-)) - G(Y_{n-1}(s-)))\mathrm{d}W(s)$$

$$I_3 = \int_{(0,t]}\int_{E_0} (K(Y_n(s-),\xi) - K(Y_{n-1}(s-),\xi))\mathrm{d}\widehat{\pi}(s,\xi).$$

Take $|.|_H^2$ on both sides, then sup over $t \in [0,T]$ and expectation

$$\begin{aligned}\mathbf{E}[\sup_{t\in[0,T]}|I_1(t)|_H^2] &\leq T\mathbf{E}\int_0^T |F(Y_n(s-)) - F(Y_{n-1}(s-))|_H^2\mathrm{d}s \\ &\leq CT\mathbf{E}\int_0^T |Y_n(s-) - Y_{n-1}(s-)|_H^2\mathrm{d}s.\end{aligned}$$

Similar estimates for $\mathbf{E}[\sup_{t\in[0,T]}|I_j(t)|_H^2]$, $j = 2,3$ using the BDG inequality:

$$\begin{aligned}\mathbf{E}[\sup_{t\in[0,T]}|I_2(t)|_H^2] &\leq c_2\mathbf{E}\int_0^T \|G(Y_n(s-)) - G(Y_{n-1}(s-))\|_{L_2(U_0,H)}^2\mathrm{d}s \\ &\lesssim \mathbf{E}\int_0^T |Y_n(s-) - Y_{n-1}(s-)|_H^2\mathrm{d}s.\end{aligned}$$

And,

$$\begin{aligned}\mathbf{E}[\sup_{t\in[0,T]}|I_3(t)|_H^2] &\lesssim \mathbf{E}\int_{(0,T]}\int_{E_0} |K(Y_n(s-),\xi) - K(Y_{n-1},\xi)|_H^2\mathrm{d}\pi(s,\xi) \\ &= \mathbf{E}\int_0^T\int_{E_0} |K(Y_n(s-),\xi) - K(Y_{n-1},\xi)|_H^2\mathrm{d}\nu(\xi)\mathrm{d}s \\ &\lesssim \mathbf{E}\int_0^T |Y_n(s-) - Y_{n-1}(s-)|_H^2\mathrm{d}s.\end{aligned}$$

Putting everything together:

$$\mathbf{E}[\sup_{t\in[0,T]}|Y_{n+1}(s) - Y_n(s)|_H^2] \leq C(T)\int_0^T \mathbf{E}|Y_n(s-) - Y_{n-1}(s-)|_H^2\mathrm{d}s \tag{7.9}$$

Observe that using the Ito isometries we have,

$$\begin{aligned}\mathbf{E}\sup_{r\in[0,s]}|Y_1(r) - Y_0|_H^2 \lesssim\ & s^2\mathbf{E}\left[\sup_{r\in[0,s]}|F(r,Y_0)|_H^2\right] + \mathbf{E}\Big|\int_0^s G(r,Y_0)\mathrm{d}W(r)\Big|_H^2 \\ &+ \mathbf{E}\Big|\int_{(0,s]}\int_{E_0} K(r,Y_0,\xi)\mathrm{d}\widehat{\pi}(r,\xi)\Big|_H^2\end{aligned}$$

$$\leq s^2(1+\mathbf{E}|Y_0|_H^2)+\mathbf{E}\int_0^s \|G(r,Y_0)\|^2_{L_2(U_0,H)}+\mathbf{E}\int_0^s\int_{E_0}|K(r,Y_0,\xi)|_H^2\mathrm{d}\nu(\xi)\mathrm{d}r$$
$$\lesssim 1+\mathbf{E}|Y_0|_H^2.$$

Iterating (7.9):

$$\mathbf{E}\Big(\sup_{0\leq t\leq T}|Y_{n+1}(t)-Y_n(t)|_H^2\Big)\leq C\int_0^T\int_0^{t_1}\cdots\int_0^{t_{n-1}}\mathbf{E}|Y_1(t_n)-Y_0|_H^2\mathrm{d}t_n\cdots\mathrm{d}t_2\mathrm{d}t_1$$

$$\Rightarrow \mathbf{E}\Big(\sup_{0\leq t\leq T}|Y_{n+1}(t)-Y_n(t)|_H^2\Big)\lesssim c^n(1+\mathbf{E}|Y_0|_H^2)\frac{T^n}{n!}, \tag{7.10}$$

and hence

$$\|Y_{n+1}-Y_n\|_{L^2(\Omega;L^\infty([0,T];H))}\lesssim\frac{(T)^{n/2}}{\sqrt{n!}},$$

where the hidden constant does not depend on n. The right-hand side of the estimate above is summable in n. Thus the sequence $(Y_n)_{n=1}^\infty$ is Cauchy in $L^2(\Omega;L^\infty([0,T];H))$. Let $Y=\lim Y_n$ in $L^2(\Omega;L^\infty([0,T];H))$. This also shows that for a fixed $t\in[0,T]$, the convergence $Y_n(t)\to Y(t)$ also holds in $L^2(\Omega;H)$.

We need to show that Y is adapted, càdlàg , and solves (7.7). We claim that $Y_n\to Y$ uniformly on $[0,T]$ as $n\to\infty$, $\mathbf{P}$-a.s. Using Chebyshev's inequality and (7.10) we see that

$$\mathbf{P}\left[\sup_{0\leq s\leq T}|Y_{n+1}(s)-Y_n(s)|_H>\frac{1}{2^n}\right]\lesssim\frac{(4T)^n}{n!}.$$

The right-hand side of the inequality above is summable in n, so the Borel-Cantelli lemma implies that

$$\mathbf{P}\Big[\limsup_n\sup_{0\leq s\leq T}|Y_{n+1}(s)-Y_n(s)|_H>\frac{1}{2^n}\Big]=0. \tag{7.11}$$

So with probability 1 we have

$$\sup_{0\leq s\leq T}|Y_{n+1}(s)-Y_n(s)|_H\leq\frac{1}{2^n}$$

holds for all but finitely many n.

Triangle inequality $\Rightarrow$ $(Y_n)_{n=1}^\infty$ is almost surely uniformly Cauchy on $[0,T]$ $\Rightarrow$ converges to Y in H uniformly on $[0,T]$, $\mathbf{P}$-a.s. **Note:** Uniform limit of H-valued càdlàg functions is also càdlàg . (Eg. The proof involves showing that the left and the right limits of the sequence of functions are Cauchy and then using triangle inequality and uniform convergence we show that the limit is càdlàg .) Hence we see that Y càdlàg a.s.
In addition, for each fixed $t\in[0,T]$ we have (upto a subsequence) that $Y_n(t)\to Y(t)$ a.s., which implies that Y is $\mathcal{F}_t$-adapted (because a.e. limit of measurable functions is measurable).

Next: Need to show that Y satisfies (7.7) almost surely. We pass to the limit in (7.8). Using the isometric formulas and the Lipschitz growth assumption it is easy to see that each term on the right-hand side of (7.8) converges in $L^2(\Omega;H)$ to the corresponding term with Y in place of Y_{n-1}. ◻

Next, well-posedness:

Uniqueness and continuous dependence on the initial data

Proposition 7.5. *In the setup above there exists $C = C(T) > 0$ such that for any two solutions Y and Z to (7.7) with respective initial conditions $Y_0, Z_0 \in L^2(\Omega, \mathcal{F}_0, \mathbf{P}; H)$ and for all $A \in \mathcal{F}_0$ we have*

$$\mathbf{E}\Big[\mathbf{1}_A \sup_{t\in[0,T]} |Y(t) - Z(t)|_H^2\Big] \leq C\mathbf{E}[\mathbf{1}_A|Y_0 - Z_0|_H^2]. \tag{7.12}$$

Remark: *Taking $A = \{Y_0 = Z_0\}$, we get a.s. uniqueness:* $\mathbf{P}[\mathbf{1}_{\{Y_0=Z_0\}}|Y(t)-Z(t)|_H = 0 \text{ for all } t \in [0,T]] = 1$, *i.e., pathwise uniqueness holds for equation (7.7).*

Proof. Very much like before + Gronwall. We will work out the case $Y_0 = Z_0$ a.e. The more general case is no different since $A \in \mathcal{F}_0$ does not change predictability of any integrals.

$$\begin{aligned}\mathbf{E}\Big[\sup_{t\in[0,T]} |Y(t) - Z(t)|_H^2\Big] &\lesssim \mathbf{E}\Big[\sup_{t\in[0,T]} \int_0^t |F(s, Y(s)) - F(s, Z(s))|_H^2 \mathrm{d}s\Big] \\ &+ \mathbf{E}\Big[\sup_{t\in[0,T]} \Big|\int_0^t [G(s, Y(s-)) - G(s, Z(s-))]\mathrm{d}W(s)\Big|_H^2\Big] \\ &+ \mathbf{E}\Big[\sup_{t\in[0,T]} \Big|\int_{(0,t]} \int_{E_0} [K(s, Y(s-), \xi) - K(s, Z(s-), \xi)]\mathrm{d}\widehat{\pi}(s, \xi)\Big|_H^2\Big] \\ &:= J_1 + J_2 + J_3.\end{aligned} \tag{7.13}$$

Estimates like earlier give us that RHS $J_1 + J_2 + J_3 \lesssim \int_0^T \mathbf{E}[\sup_{s\in[0,t]} |Y(s) - Z(s)|_H^2]\mathrm{d}t$.

$$\Rightarrow \mathbf{E}[\sup_{t\in[0,T]} |Y(t) - Z(t)|_H^2] = 0. \tag{7.14}$$

□

Next we will apply the Itô formula to the solution of (7.7) with the norm in L^p. Recall Itô formula: Let $\psi : H \to \mathbb{R}$ s.t. ψ is C^2

$$\begin{aligned}\psi(Y(t)) = \ &\psi(Y_0) + \int_0^t (D\psi(Y(s-)), \mathrm{d}Y(s))_H \mathrm{d}s + \frac{1}{2}\int_0^t \mathrm{Tr}[D^2\psi(Y(s-))G(Y(s-))G^*(Y(s-))]\mathrm{d}s \\ + &\int_{(0,t]} \int_E [\psi(Y(s-) + K(s, \xi)) - \psi(Y(s-))]\,\mathrm{d}\pi(s, \xi) - \int_{(0,t]} \int_E \langle D\psi(Y(s-)), K(s, \xi)\rangle_H \mathrm{d}\pi(s, \xi).\end{aligned}$$

Frechet derivatives:

Take $\psi(x) = |x|_H^p$. Consider $g(x) = |x|_H^2$ then $Dg(x) = 2 < x, \cdot >_H$. For $\psi(x) = g(x)^{\frac{p}{2}}$. Then

$$D\psi : H \mapsto H'; \quad D\psi(x) = \frac{p}{2} g(x)^{\frac{p}{2}-1} 2 < x, \cdot >_H = pg(x)^{\frac{p}{2}-1} x = p|x|_H^{p-2} x.$$

And $D^2\psi : H \mapsto L(H, H')$;

$$D^2\psi(x) = pD\left(|x|_H^{p-2}\right) x + p|x|_H^{p-2} Dx = p(p-2)|x|_H^{p-4} < x, \cdot >_H x + p|x|_H^{p-2} I.$$

So far: For $Y_0 \in L^2(\Omega; H)$ we have shown $\exists!\, Y \in L^2(\Omega; L^\infty(0, T; H))$.

Next: For $Y_0 \in L^p(\Omega; H) \Rightarrow \exists!\, Y \in L^p(\Omega; L^\infty(0, T; H))$. For this we would additionally assume that for some $c > 0$

$$\int_{E_0} |K(x,\xi)|_H^p \mathrm{d}\nu(\xi) \le c(1+|x|_H^p) \quad \forall x \in H. \tag{7.15}$$

We apply Itô formula,

$$\begin{aligned}\langle D\psi(Y(s-)), \mathrm{d}Y(s)\rangle_H &= \langle p|Y(s-)|_H^{p-2}Y(s-), F(Y(s-))\mathrm{d}s + G(Y(s-))\mathrm{d}W(s) \\ &\quad + \int_{E_0} K(Y(s-),\xi)\mathrm{d}\widehat{\pi}(s,\xi)\rangle_H,\end{aligned}$$

$$\begin{aligned}Tr[D^2\psi(Y(s-))G(Y(s-))G(Y(s-))^*] &= p(p-2)|Y(s-)|_H^{p-4}Tr[\langle Y(s-), GG^*\rangle_H Y(s-)] \\ &\quad + p|Y(s-)|_H^{p-2}Tr[GG^*] \\ &=: p(p-2)|Y(s-)|_H^{p-4}T_1 + p|Y(s-)|_H^{p-2}T_2.\end{aligned}$$

where,

$$\begin{aligned}T_1 &= \sum_{k=1}^{\infty}\langle Y(s-), GG^*e_k\rangle_H\langle Y(s-), e_k\rangle_H = \sum_{k=1}^{\infty}\langle GG^*Y(s-), e_k\rangle_H\langle Y(s-), e_k\rangle_H \\ &= \langle GG^*Y(s-), Y(s-)\rangle_H = |G^*Y(s-)|_{U_0}^2.\end{aligned}$$

We obtain

$$Tr[D^2\psi(Y(s-))G(Y(s-))G(Y(s-))^*] = p(p-2)|Y(s-)|_H^{p-4}|G^*Y(s-)|_{U_0}^2 + p|Y(s-)|_H^{p-2}\|G\|_{L_2(U_0;H)}^2.$$

Collect terms:

$$\begin{aligned}&|Y(t)|_H^p = |Y_0|_H^p + \int_0^t p|Y(s-)|_H^{p-2}\langle Y(s-), F(Y(s-))\rangle_H \mathrm{d}s + \\ &\int_0^t p|Y(s-)|_H^{p-2}\langle Y(s-), G(Y(s-))\mathrm{d}W(s)\rangle_H + \int_{(0,t]}\int_{E_0} p|Y(s-)|_H^{p-2}\langle Y(s-), K(Y(s-),\xi)\rangle_H \mathrm{d}\widehat{\pi}(s,\xi) \\ &+ \frac{1}{2}\int_0^t p|Y(s-)|_H^{p-2}\|G(Y(s-))\|_{L_2(U_0;H)}^2 + p(p-2)|Y(s-)|^{p-2}|G(Y(s-))^*Y(s-)|_{U_0}^2 \mathrm{d}s \\ &+ \int_{(0,t]}\int_{E_0} |Y(s-)+K(Y(s-),\xi)|_H^p - |Y(s-)|_H^p - p|Y(s-)|_H^{p-2}\langle Y(s-), K(Y(s-),\xi)\rangle_H \mathrm{d}\pi(s,\xi) \\ &=: I_1 + I_2 + I_3 + I_4 + I_4.\end{aligned}$$

8. Week 8

(Continued..)Now we find estimates for the above terms:

$$\begin{aligned}\mathbf{E}[\sup_{s\in[0,T]} I_1(s)] &\le p\mathbf{E}\sup_{t\in[0,T]}\int_0^t |Y(s-)|_H^{p-1}F(Y(s-))|_H \mathrm{d}s \\ &\lesssim \mathbf{E}\int_0^T p|Y(s-)|_H^{p-1}(1+|Y(s-)|_H)\mathrm{d}s \\ &\lesssim 1 + \int_0^T \mathbf{E}(\sup_{r\in(0,s)}|Y(r-)|_H^p)\mathrm{d}s.\end{aligned}$$

$$\mathbf{E}[\sup_{t\in[0,T]} I_2(t)] \leq_{\mathrm{BDG}} \mathbf{E}\int_0^T |Y(s-)|^{2(p-2)}\|\langle G(Y(s-))^*Y(s-),Y(s-)\rangle_H\|^2_{L_2(U_0,H)}$$

$$\leq \mathbf{E}\left(\int_0^T |Y(s-)|_H^{2p-4}|G(Y(s-))^*Y(s-)|_{U_0}\mathrm{d}s\right)^{\frac{1}{2}}$$

$$\leq \mathbf{E}\left(\int_0^T |Y(s-)|_H^{2p-4}|Y(s-)|_H^2\|G(Y(s-))^*\|^2_{L_2(H,U_0)}\mathrm{d}s\right)^{\frac{1}{2}}$$

$$= \mathbf{E}[(\sup_{s\in(0,T]}|Y(s-)|_H^p)^{\frac{1}{2}}\left(\int_0^T |Y(s-)|_H^{p-2}\|G(Y(s-))\|^2_{L_2(U_0,H)}\mathrm{d}s\right)^{\frac{1}{2}}]$$

Using Cauchy-Schwarz and the growth assumptions

$$\leq \frac{1}{4}\mathbf{E}(\sup_{s\in[0,T]}|Y(s-)|_H^p) + \mathbf{E}\left(\int_0^T 1+|Y(s-)|_H^p\mathrm{d}s\right)$$

$$\leq \frac{1}{4}\mathrm{LHS} + C(1+\int_0^T \mathbf{E}(\sup_{r\in(0,s)}|Y(r-)|_H^p)\mathrm{d}s).$$

Same estimate for $I_{3,4,5}$. We apply the BDG inequality (Proposition 5.6) to I_4 and obtain

$$\mathbf{E}\Big[\sup_{s\in[0,t]}|I_4(s)|_H\Big] \lesssim \mathbf{E}\Big(\int_{(0,t]}\int_{E_0}(p|Y(s-)|_H^{p-2}(Y(s-),K(s,Y(s-),\xi)))^2\mathrm{d}\pi(s,\xi)\Big)^{1/2}$$

$$\lesssim \mathbf{E}\Big(\int_{(0,t]}\int_{E_0}|Y(s-)|_H^{2p-2}|K(s,Y(s-),\xi)|_H^2\mathrm{d}\pi(s,\xi)\Big)^{1/2}$$

$$\leq \mathbf{E}\Big(\sup_{s\in[0,t]}|Y(s)|_H^{p/2}\left(\int_{(0,t]}\int_{E_0}|Y(s-)|_H^{p-2}|K(s,Y(s-),\xi)|_H^2\mathrm{d}\pi(s,\xi)\right)^{1/2}\Big).$$

Then we use Young's inequality and growth assumption as above. In the estimate of I_5 we make use of the inequality

$$\Big||x+y|_H^p-|x|_H^p-p|x|_H^{p-2}(x,y)_H\Big| \lesssim (|x|_H^{p-2}|y|_H^2+|y|_H^p) \qquad \text{for all } x,y\in H, \tag{8.1}$$

Thus we obtain

$$\mathbf{E}\Big[\sup_{s\in[0,t]}|I_5(s)|_H\Big] \leq \mathbf{E}\int_{(0,t]}\int_{E_0}\Big||Y(s-)+K(s,Y(s-),\xi)|_H^p-|Y(s-)|_H^p$$

$$-p|Y(s-)|_H^{p-2}(Y(s-),K(s,Y(s-),\xi))_H\Big|\mathrm{d}\pi(s,\xi)$$

$$\lesssim \mathbf{E}\int_{(0,t]}\int_{E_0}\left[|Y(s-)|_H^{p-2}|K(s,Y(s-),\xi)|_H^2+|K(s,Y(s-),\xi)|_H^p\right]\mathrm{d}\pi(s,\xi)$$

$$= \mathbf{E}\int_{(0,t]}\int_{E_0}\left[|Y(s-)|_H^{p-2}|K(s,Y(s-),\xi)|_H^2+|K(s,Y(s-),\xi)|_H^p\right]\mathrm{d}\nu(\xi)\mathrm{d}s. \tag{8.2}$$

Using (7.15) and Young's inequality as above we obtain

$$\mathbf{E}\Big[\sup_{s\in[0,t]}|I_5(s)|_H\Big] \lesssim 1+\mathbf{E}\int_0^t|Y(s-)|_H^p\mathrm{d}s. \tag{8.3}$$

Combining all the terms we obtain

$$\frac{1}{2}\mathbf{E}(\sup_{s\in[0,t]}|Y(s-)|_H^p) \leq \mathbf{E}[|Y_0|_H^p] + C\int_0^t \mathbf{E}(\sup_{r\in[0,s]}|Y(r-)|_H^p)\mathrm{d}s$$

We have the form, $z(t) \leq C + C\int_0^t z(s)\mathrm{d}s$. We will apply the Gronwall inequality:

$$\mathbf{E}(\sup_{s\in[0,T]} |Y(s-)|_H^p) \leq C(1+\mathbf{E}|Y_0|_H^p). \tag{8.4}$$

8.1. The main SPDE.

$$\begin{aligned} \mathrm{d}u + [Au(s) &+ B(u(s),u(s)) + F(u(s))]\mathrm{d}s \\ &= G(u(s-))\mathrm{d}W(s) + \int_{E_0} K(u(s-),\xi)\mathrm{d}\widehat{\pi}(s,\xi) + \int_{E\setminus E_0} \mathscr{K}(u(s-),\xi)\mathrm{d}\pi(s,\xi). \\ u(0) &= u_0. \end{aligned} \tag{8.5}$$

8.1.1. *Functional Framework.* Hilbert spaces $V \subset H = H' \subset V'$, dense, compact embedding. Notation: V: $((\cdot,\cdot))$ and H: $(\cdot,\cdot)$.

Assumptions on A:

(1) A is linear unbounded operator $A : D(A) \subset V \to H$
(2) $D(A)$ is dense in V.
(3) A is bijective from $D(A)$ onto H.
(4) $(Au,u) = ((u,v))$ for all $u \in D(A), v \in V$.

Properties of A:

(1) $D(A)$ is complete under the norm $|u|_{D(A)=|A(u)|}$ for all $u \in D(A)$.
(2) A is symmetric $(Au,v) = (u,Av)$ for all $u,v \in D(A)$.
(3) $A : V \to V'$ is continuous. For $u \in V$ define $Au \in V'$ by $(Au,v) = ((u,v))$ for all $v \in V$ and $|Au|_{V'} = \|u\|$.
(4) $A^{-1} : H \to D(A) \subset V \subset H$ continuous.
(5) $A^{-1} : H \to V \to H$ is a compact operator.
(6) A^{-1} is self-adjoint (A is symmetric).
(7) A^{-1} is positive definite: $\forall u,v \in H, (A^{-1}u,u) = (A^{-1}u, AA^{-1}u) = \|A^{-1}u\|^2 > 0$.

Then the spectral theorem implies that there exists an ONB $(u_k)_1^\infty$ of H consisting of eigenvectors of A^{-1}, with eigenvalues $(\mu_k)_1^\infty$ s.t. $\mu_k \geq 0$ and $\lambda_k \to 0$ as $k \to \infty$. Then $w_k = A^{-1}u_k$ is an eigenvector for A because $Aw_k = u_k = \frac{1}{\mu_k}w_k$. Then $w_k \in D(A)$ and the w_k's form an ONB in H with eigenvalues $\lambda_k \geq 0$ s.t. $\lambda_k \to \infty$ as $k \to \infty$.

Assumptions on B:

(1) $B : V \times D(A) \to V'$ bilinear continuous.

$$|(B(v,u),w)| \leq C\|v\||Au|\|w\|.$$

(2) $B : D(A) \times D(A) \to H$ bilinear continuous.

$$|(B(v,u),w)| \leq C_0\|v\|^{\frac{1}{2}}|Av|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}|Au|^{\frac{1}{2}}|w|. \tag{8.6}$$

(3) Cancellation property for B: $(B(v,u),u) = 0$ for all $v \in V, u \in D(A)$.

Assumptions on F: $F : V \to H$ linear growth and is Lipschitz. $|F(u)| \leq c(1+\|u\|)$ and $|F(u)-F(v)| \leq c\|u-v\|$.

Noise terms:

(1) W is a Wiener process with RKHS (reproducing Kernel Hilbert space) U_0. $G: H \to L_2(U_0, H)$, $G: V \to L_2(U_0, V)$. We assume G has linear growth and is Lipschitz w.r.t both the H-norm and the V-norm.

(2) π is the jump measure of a Lévy process L ($\pi = \sum_{s=0}^{\infty} \delta_{s,\Delta L(s)}$, sum is finite).
$\mathscr{K} : H \times E \to V$, E_0 = unit ball in E s.t.

$$\int_{(0,t]} \int_{E \setminus E_0} \mathscr{K}(u(s-), \xi) \mathrm{d}\pi(s, \xi) = \sum_{s \in (0,t], \Delta L(s) \neq 0} \mathscr{K}(u(s-), \Delta L(s)).$$

(3) $K : H \times E_0 \to H$, $K : V \times E_0 \to H$. We assume that K has linear growth and is Lipschitz in the sense that

$$\int_{E_0} |K(v, \xi)|^2 \mathrm{d}\nu(\xi) \leq c(1 + |v|^2)$$

where ν is the Lévy measure of the Lévy process L. Same with the V-norm + Lipschitz condition in V +

$$\int_{E_0} \|K(v, \xi)\|^4 \mathrm{d}\nu(\xi) \leq c(1 + \|v\|^4) \quad \forall v \in V.$$

8.1.2. *Strategy for proving existence.* First we ignore the $\mathrm{d}\pi$ term (i.e. set $\mathscr{K} = 0$) and consider

$$\begin{aligned} \mathrm{d}u + [Au(s) + B(u(s), u(s)) + F(u(s))]\mathrm{d}s &= G(u(s-))\mathrm{d}W(s) + \int_{E_0} K(u(s-), \xi)\mathrm{d}\widehat{\pi}(s, \xi). \\ u(0) &= u_0. \end{aligned} \tag{8.7}$$

Now we consider a truncation of the B-term:

$$\begin{aligned} \mathrm{d}u + [Au(s) + \theta(\|u - u_*\|)B(u(s), u(s)) &+ F(u(s))]\mathrm{d}s \\ &= G(u(s-))\mathrm{d}W(s) + \int_{E_0} K(u(s-), \xi)\mathrm{d}\widehat{\pi}(s, \xi). \\ u(0) &= u_0. \end{aligned} \tag{8.8}$$

where u_* solves the linearized random equations:

$$\begin{aligned} \frac{\mathrm{d}u_*}{\mathrm{d}t} + Au_* &= 0 \\ u_*(0) &= u_0. \end{aligned} \tag{8.9}$$

$\theta : [0, \infty) \to [0, 1]$ with a specific choice of $R < \frac{3}{C_0}$ where C_0 is from (8.6). If u solves (8.8) then u solves (8.7) until the first time that $\|u - u_*\| > \frac{R}{2}$.

Definition 8.1. *A pair $(\widetilde{S}, \widetilde{u})$ is a martingale solution to (8.8) if*

- $\widetilde{S} = (\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t \geq 0}, \widetilde{\mathbf{P}}, \widetilde{W}, \widetilde{\pi})$.
- $\widetilde{u} \in L^4(\widetilde{\Omega}, L^\infty(0, T; V)) \cap L^4(\widetilde{\Omega}, L^2(0, T; D(A)))$.
- *$\widetilde{u}$ has càdlàg sample paths in H a.s. and $\widetilde{u}$ is $\widetilde{\mathcal{F}}_t$-adapted.*
- *and $\forall t \in [0, T]$, $\widetilde{\mathbf{P}}$-a.s. we have*

$$\begin{aligned} \widetilde{u}(t) + \int_0^t [A\widetilde{u}(s) + \widetilde{\theta}(u(s) - u_*(s))B(\widetilde{u}(s), \widetilde{u}(s)) + F(\widetilde{u}(s))]\mathrm{d}s \\ = \widetilde{u}_0 + \int_0^t G(\widetilde{u}(s-))\mathrm{d}\widetilde{W}(s) + \int_{(0,t]} \int_{E_0} K(\widetilde{u}(s-), \xi)\mathrm{d}\widehat{\widetilde{\pi}}(s, \xi). \end{aligned} \tag{8.10}$$

and $\widetilde{u}(0)$ has the same law as u_0.

9. Week 9

9.0.1. *Galerkin scheme and estimates.* We use $\{w_k\}_{k=1}^{\infty}$ as a basis for the Galerkin method. $H_n = \text{span}\{w_1, ..., w_n\}$. Let P_n be the projection from H onto H_n. Consider the H_n valued SDE

$$\begin{aligned} &\mathrm{d}u^n + [Au^n + \theta(\|u^n - u^n_*\|)P_nB(u^n,u^n) + P_nF(u^n)] \\ &\qquad = P_nG(u^n)\mathrm{d}W + \int_{E_0} P_nK(u^n(t-),\xi)\mathrm{d}\widehat{\pi}(s,\xi) \\ &u^n(0) = P_nu_0 = u^n_0. \end{aligned} \tag{9.1}$$

And u^n_* is the solution of the linearized equation (8.9) wuth $u^n_*(0) = u^n_0$. We have the existence and uniqueness of u^n solving (9.1) globally that is on $(0,T)$ (thanks to the truncation on B and to the hypotheses on the growth).

Estimates: In order to make estimates on u^n we need some estimates on u^n_*.

Lemma 9.1. *$\forall p \in (1,\infty)$, $\exists C_p > 0$ s.t. $\forall u_0 \in L^p(\Omega, V)$ we have*

$$\sup_{s\in(0,t]} \|u^n_*(s)\|^p + \int_0^t \|u^n_*(s)\|^{p-2}|Au^n_*(s)|^2\mathrm{d}s + (\int_0^t |Au^n_*(s)|^2\mathrm{d}s)^{\frac{p}{2}} \leq C_p(1 + \|u_0\|^p)$$

for all n, for all $t \in [0,T]$, $\mathbf{P}$-a.s.

Proposition 9.2. *Let $u_0 \in L^4(\Omega, V)$. Then there exists $C = C(T) > 0$ s.t. $\forall n$*

$$\sup_n \mathbf{E}[\sup_{s\in[0,T]} \|u^n(s)\|^4] \leq C(1 + \mathbf{E}\|u_0\|^4) \tag{9.2}$$

$$\sup_n \mathbf{E}(\int_0^T |Au^n(s)|^2\mathrm{d}s)^2 \leq C(1 + \mathbf{E}\|u_0\|^8) \tag{9.3}$$

$$\sup_n \mathbf{E}(\int_0^T \|u^n(s)\|^2|Au^n(s)|^2\mathrm{d}s) \leq C(1 + \mathbf{E}\|u_0\|^8) \tag{9.4}$$

Proof. (i) Let $v^n = u^n - u^n_*$, then v^n satisfies

$$\mathrm{d}v^n + [Av^n + \theta(\|v^n\|)P_nB(u^n,u^n) + P_nF(u^n)] = P_nG(u^n)\mathrm{d}W + \int_{E_0} P_nK(u^n(t-),\xi)\mathrm{d}\widehat{\pi}(s,\xi)$$
$$v^n(0) = 0.$$

Observe that (8.9) implies $\|u^n_*(t)\| \leq C\|u^n_0\| \leq C\|u_0\|$ and $\|u^n\|^4 \leq \|v^n\|^4 + \|u^n_*\|^4$. So we need to find estimates for $\|v^n\|$. For that we apply Itô's formula with $\psi(v) = \|v\|^p$ with $p = 4$.

$$\begin{aligned} &\|v^n\|^4 + 4\int_0^t \|v^n(s)\|^2[((Av^n(s),v^n(s))) + \theta(\|v^n(s)\|)((P_nB(u^n(s),u^n(s)),v^n(s))) + ((P_nF(u^n(s)),v^n(s)))] \\ &= 4\int_0^t \|v^n(s-)\|^2((v^n(s),P_nG(u^n(s-))\mathrm{d}W(s))) + \int_{(0,T]}\int_{E_0} 4\|v^n(s-)\|^2((v^n(s-),P_nK(u^n(s-),\xi)))\mathrm{d}\widehat{\pi}(s,\xi) \\ &+ \frac{1}{2}\int_0^t p\|v^n(s-)\|^2\|P_nG(u^n(s-))\|^2_{L_2(U_0,V)} + 8|[P_nG(u^n(s-))]^*v^n(s-)|^2_{U_0}\mathrm{d}s \\ &+ \int_{(0,t]}\int_{E_0} [\|v^n(s-) + P_nK(u^n(s-),\xi)\|^4 - \|v^n(s-)\|^4 - 4\|v^n(s-)\|^2((v^n(s-),P_nK(u^n(s-),\xi)))]\mathrm{d}\pi(s,\xi) \\ &= I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

First consider $I_1 = 4\int_0^t \|v^n\|^2 (P_n F u^n(s), v^n(s))$. Then,

$$\begin{aligned}
\mathbf{E} \sup_{t\in[0,T]} |I_1(t)| &\leq 4\mathbf{E}\int_0^T \|v^n(s)\|^2 |P_n F(u^n(s))||Av^n(s)| \mathrm{d}s \\
&\leq 4\mathbf{E}\int_0^T \|v^n(s)\|^2 |F(u^n(s))||Av^n(s)| \mathrm{d}s \\
&\leq 4\mathbf{E}\int_0^T \|v^n(s)\|^2 (1+\|u^n(s)\|)|Av^n(s)| \mathrm{d}s \\
&\leq \mathbf{E}\int_0^T \|v^n(s)\|^2 |Av^n(s)|^2 + \mathbf{E}\int_0^T \|v^n(s)\|^2 (1+\|u^n(s)\|)^2 \mathrm{d}s
\end{aligned}$$

Next,

$$\mathbf{E} \sup_{t\in[0,T]} |I_2(t)| \leq 4\mathbf{E}\int_0^T \theta(\|v^n(s)\|)\|v^n(s)\|^2 (B(u^n(s)), Av^n(s)) \mathrm{d}s$$

We write

$$\begin{aligned}
(B(u^n(s)), Av^n(s)) &= (B(v^n(s)+u_*^n(s)), Av^n(s)) \\
&\leq (B(v^n(s), v^n(s)) + B(v^n(s), u_*^n(s)) + B(u_*^n(s), v^n(s)) + B(u_*^n(s), u_*^n(s)), Av^n(s)) \\
&=: J_1 + J_2 + J_3 + J_4
\end{aligned}$$

Next,

$$\begin{aligned}
|J_1| &\leq c_0 \|v^n\| |Av^n|^2, \\
|J_2| + |J_3| &\leq c_0 \|u_*^n\|^{\frac{1}{2}} |Au_*^n|^{\frac{1}{2}} \|v^n\|^{\frac{1}{2}} |Av^n|^{\frac{3}{2}} \\
&\leq |Av^n|^2 + C\|v^n\|^2 \|u_*^n\|^2 |Au_*^n|^2 \\
|J_4| &\leq \|u_*^n\| |Au_*^n| |Av^n| \\
&\leq |Av^n|^2 + \|u_*^n\|^2 |Au_*^n|^2.
\end{aligned}$$

This implies

$$\begin{aligned}
|I_2| &\leq \int_0^t c_0\theta(\|v^n\|)\|v^n\|^3 |Av^n|^2 + \|v^n\|^2 \theta(\|v^n\|)(|Av^n|^2 + C\|u_*^n\|^2 |Au_*^n|^2 (1+\|v^n\|^2)) \mathrm{d}s \\
&\leq \int_0^t c_0 R\|v^n\|^2 |Av^n|^2 + \|v^n\|^2 |Av^n|^2 + CR^2(1+R^2)\|u_*^n\|^2 |Au_*^n|^2 \mathrm{d}s
\end{aligned}$$

Then taking R small enough and combining the rest of the terms we obtain

$$\begin{aligned}
&\mathbf{E}[\sup_{t\in[0,T]} \|v^n\|^4] + \mathbf{E}\int_0^T \|v^n(s)\|^2 |Av^n(s)|^2 \mathrm{d}s \\
&\quad \leq \mathbf{E}\int_0^T (1+\|u^n(s)\|^4)\mathrm{d}s + \mathbf{E}\int_0^T \|v^n(s)\|(1+\|u^n(s)\|^2)\mathrm{d}s + \mathbf{E}\int_0^T \|u_*^n(s)\|^2 |Au_*^n(s)|^2 \mathrm{d}s
\end{aligned}$$

Terms I_3 and I_4 are treated like last week. Next,

$$\begin{aligned}
\mathbf{E}[\sup_{s\in[0,T]} |I_5(s)|] &= \mathbf{E}\int_{(0,T]}\int_{E_0} \Big| \|v^n(s-) + P_n K(u^n(s-),\xi)\|^4 - \|v^n(s-)\|^4 \\
&\qquad - 4\|v^n(s-)\|^2 ((v^n(s-), P_n K(u^n(s-),\xi))) \Big| \mathrm{d}\pi(s,\xi) \\
&\leq C\mathbf{E}\int_{(0,T]}\int_{E_0} \Big[\|v^n(s-)\|^2 \|P_n K(u^n(s-),\xi)\|^2 + \|P_n K(u^n(s-),\xi)\|^4 \Big] \mathrm{d}\pi(s,\xi)
\end{aligned}$$

$$= C\mathbf{E}\int_0^T\int_{E_0}\left[\|v^n(s-)\|^2\|K(u^n(s-),\xi)\|^2+\|K(u^n(s-),\xi)\|^4\right]\mathrm{d}\nu(\xi)\mathrm{d}s$$
$$\leq C\mathbf{E}\int_0^T\left[1+\|u^n_*(s)\|^4+\|v^n(s)\|^4\right]\mathrm{d}s.$$

Combining $I_1 - I_5$ we find that there exists a constant $C = C(R,T) > 0$ such that

$$\mathbf{E}\left[\sup_{s\in[0,T]}\|v^n(s)\|^4\right]+\int_0^T\|v^n(s)\|^2|Av^n(s)|^2\mathrm{d}s$$
$$\leq C\int_0^t\mathbf{E}\left[1+\|u^n_*(s)\|^2|Au^n_*(s)|^2+\left(\sup_{r\in[0,T]}\|v^n(r)\|^4\right)+\|u^n_*(s)\|^4\right]\mathrm{d}s. \tag{9.5}$$

Using Lemma 9.1 and Gronwall we conclude that there exists a constant $C = C(R,T) > 0$ that does not depend on n such that

$$\mathbf{E}\left[\sup_{s\in[0,T]}\|v^n(s)\|^4\right]+\mathbf{E}\int_0^T\|v^n(s)\|^2|Av^n(s)|^2\mathrm{d}s\leq C(1+\mathbf{E}\|u_0\|^4). \tag{9.6}$$

So we have now shown that $\{u^n\}_{n=1}^{\infty}$ are bounded in $L^4(\Omega;L^\infty(0,T;V))$. Next we will show that $\{u^n\}_{n=1}^{\infty}$ are bounded in $L^4(\Omega;L^2(0,T;D(A)))$. Apply Ito's formula to the equation with v^n using the function $\|v\|^2$.

$$\|v^n(T)\|^2+2\int_0^T|Av^n(s)|^2\mathrm{d}s+2\underbrace{\int_0^T\theta(\|v^n(s)\|)(B(u^n(s),u^n(s)),Av^n(s))\mathrm{d}s}_{J_{1,1}}+\underbrace{2\int_0^T(Fu^n(s),Av^n(s))\mathrm{d}s}_{J_{1,2}}$$

$$=2\int_0^T((G(u^n(s-),v^n(s))))\mathrm{d}W(s)+2\int_{(0,T]}\int_{E_0}((v^n(s),K(u^n(s-),\xi)))\mathrm{d}\widehat{\pi}(s,\xi)$$
$$+\int_0^T\|P_nG(u^n(s))\|^2_{L_2(U,V)}\mathrm{d}s+\int_{(0,T]}\int_{E_0}\|P_nK(u^n(s-),\xi)\|^2\mathrm{d}\pi(s,\xi)=:J_1+...+J_4$$

Like earlier we can show that for any $\varepsilon>0$, $J_{1,2}\leq\int_0^T(\varepsilon|Av^n(s)|^2+C_\varepsilon(1+\|u^n(s)\|^2))\mathrm{d}s$ and $J_{1,1}\leq\int_0^T(\varepsilon|Av^n(s)|^2+C_\varepsilon(\|u^n_*(s)\|^2|Au^n_*|^2))\mathrm{d}s$ Next,

$$\mathbf{E}[|J_3(t)|^2]=4\mathbf{E}\int_0^t\int_{E_0}|((v^n(s-),K(u^n(s-),\xi)))|^2\mathrm{d}\nu(\xi)\mathrm{d}s$$
$$\leq 4\mathbf{E}\int_0^t\int_{E_0}\|v^n(s-)\|^2\|K(u^n(s-),\xi)\|^2\mathrm{d}\nu(\xi)\mathrm{d}s$$
$$\leq C\mathbf{E}\int_0^t\|v^n(s)\|^2(1+\|u^n(s)\|^2)\mathrm{d}s$$
$$\leq C\mathbf{E}\int_0^t\left[1+\|u^n_*(s)\|^4+\|v^n(s)\|^4\right]\mathrm{d}s, \tag{9.7}$$

Observe that $f(s,\xi):=\|P_nK(u^n(s-),\xi)\|^2$ belongs to the space $\mathbf{F}^1_{\nu,T}(H)\cap\mathbf{F}^2_{\nu,T}(H)$. And thus we can write

$$J_4(t)=\int_{(0,t]}\int_{E_0}\|P_nK(u^n(s-),\xi)\|^2\mathrm{d}\widehat{\pi}(s,\xi)+\int_0^t\int_{E_0}\|P_nK(u^n(s-),\xi)\|^2\mathrm{d}\nu(\xi)\mathrm{d}s \tag{9.8}$$

Thus,

$$\begin{aligned}
\mathbf{E}[|J_4(t)|^2] &\le 2\mathbf{E}\int_0^t\int_{E_0}\|P_nK(u^n(s-),\xi)\|^4\mathrm{d}\nu(\xi)\mathrm{d}s \\
&\quad + 2\mathbf{E}\Big(\int_0^t\int_{E_0}\|P_nK(u^n(s-),\xi)\|^2\mathrm{d}\nu(\xi)\mathrm{d}s\Big)^2 \\
&\le C\mathbf{E}\int_0^t(1+\|u^n(s)\|^4)\mathrm{d}s + C\mathbf{E}\Big(\int_0^t(1+\|u^n(s)\|^2)\mathrm{d}s\Big)^2 \\
&\le C\mathbf{E}\int_0^t[1+\|u^n_*(s)\|^4+\|v^n(s)\|^4]\mathrm{d}s, \qquad (9.9)
\end{aligned}$$

Thus combining Lemma 9.1 and (9.2)-(9.3) we obtain

$$\begin{aligned}
\mathbf{E}\Big(\int_0^T|Av^n(s)|^2\mathrm{d}s\Big)^2 &\le C\mathbf{E}\Big(\int_0^T\|u^n_*(s)\|^2|Au^n_*(s)|^2\mathrm{d}s\Big)^2 + C\mathbf{E}\int_0^T\Big[1+\|u^n_*(s)\|^4+\|v^n(s)\|^4\Big]\mathrm{d}s \\
&\le C(1+\mathbf{E}\|u_0\|^8).
\end{aligned}$$

□

9.1. **Fractional Sobolev spaces.**

Definition 9.3. *Let X be a real, separable Hilbert space, let $1<p<\infty$ and $\alpha\in(0,1)$, then the space*

$$W^{\alpha,p}(0,T;X) := \Big\{u\in L^p(0,T;X) : \int_0^T\int_0^T\frac{|u(t)-u(s)|_X^p}{|t-s|^{1+\alpha p}}\mathrm{d}t\mathrm{d}s<\infty\Big\},$$

is a Banach space under the norm

$$\|u\|^p_{W^{\alpha,p}(0,T;X)} := \int_0^T|u(t)|_X^p\mathrm{d}t + \int_0^T\int_0^T\frac{|u(t)-u(s)|_X^p}{|t-s|^{1+\alpha p}}\mathrm{d}t\mathrm{d}s.$$

The spaces $W^{\alpha,p}(0,T;X)$ are referred to as fractional Sobolev spaces.

We want to next show that $\{u^n\}_{n=1}^\infty$ are bounded in $L^2(\Omega;W^{\alpha,2}(0,T;H))$ for any $\alpha\in(0,\frac{1}{2})$.
Preparation: There exists $C>0$ s.t. for any $\psi\in L^2_{U_0,T}(H)$, we have

$$\mathbf{E}\|\int_0^\cdot\psi(s)\mathrm{d}W(s)\|^2_{W^{\alpha,2}(0,T;H)}\le C\mathbf{E}\int_0^T\|\psi(s)\|^2_{L_2(U_0,H)}\mathrm{d}s. \qquad (9.10)$$

A proof can be found in Flandoli-Gatarek (Lemma 2.1) Martingale and stationary solution for stochastic NSE 1995.
Similarly there exists $c>0$ s.t. for all $f\in F^2_{\nu,T}(H)$ we have,

$$\mathbf{E}\|\int_{(0,T]}\int_{E_0}f\mathrm{d}\widehat{\pi}\|^2_{W^{\alpha,2}(0,T;H)}\le c\mathbf{E}\int_0^T\int_{E_0}|f(s,\xi)|^2\mathrm{d}\nu(s)\mathrm{d}s \qquad (9.11)$$

This can be found using the BDG inequality.
Now let

$$f^n(t)=u^n(t)-\int_0^tP_nG(u^n(s-))\mathrm{d}W(s)-\int_{(0,T]}\int_{E_0}P_nK(u^n(s-),\xi)\mathrm{d}\widehat{\pi}(s,\xi)$$

To show that $(u^n)_{n=1}^\infty$ is bounded in $L^2(\Omega, W^{\alpha,2}(0,T;H))$ it suffices to show that f_n is bounded in $L^2(\Omega, W^{\alpha,2}(0,T;H))$. Because growth conditions on the noise terms give us,

$$\mathbf{E}\left\|\int_0^t P_n G(u^n(s-))\mathrm{d}W(s)\right\|^2_{W^{\alpha,2}([0,T];H)} \leq C\mathbf{E}\int_0^T \|P_n G(u^n(s-))\|^2_{L_2(U_0,H)}\,\mathrm{d}s$$
$$\leq C\mathbf{E}\int_0^T (1+|u^n(s)|^2)\mathrm{d}s$$

And,

$$\mathbf{E}||\int_{(0,T]}\int_{E_0} P_n K(u^n(s-),\xi)\mathrm{d}\widehat{\pi}(s,\xi)||^2_{W^{\alpha,2}([0,T];H)} \lesssim \mathbf{E}\int_0^T\int_{E_0} \|P_n K(u^n(s-),\xi)\|^2_H\,\mathrm{d}\nu(\xi)\mathrm{d}s$$
$$\lesssim \mathbf{E}\int_0^T (1+|u^n(s)|^2)\mathrm{d}s.$$

$$\begin{aligned}\|f^n\|^2_{W^{\alpha,2}([0,T];H)} &= \left\|u_0^n - \int_0^\cdot Au^n\mathrm{d}s - \int_0^\cdot \theta(\|v^n\|)P_nB(u^n)\mathrm{d}s - \int_0^\cdot P_nF(u^n)\mathrm{d}s\right\|^2_{W^{\alpha,2}([0,T];H)}\\
&\lesssim \left\|u_0^n - \int_0^\cdot Au^n\mathrm{d}s - \int_0^\cdot \theta(\|v^n\|)P_nB(u^n)\mathrm{d}s - \int_0^\cdot P_nF(u^n)\mathrm{d}s\right\|^2_{W^{1,2}([0,T];H)}\\
&\lesssim \left\|u_0^n - \int_0^\cdot Au^n\mathrm{d}s - \int_0^\cdot \theta(\|v^n\|)P_nB(u^n)\mathrm{d}s - \int_0^\cdot P_nF(u^n)\mathrm{d}s\right\|^2_{L^2([0,T];H)}\\
&\quad + \int_0^T |Au^n|^2\mathrm{d}s + \int_0^T \theta(\|v^n\|)^2|B(u^n)|^2\mathrm{d}s + \int_0^T |F(u^n)|^2\mathrm{d}s,\\
&\lesssim |u_0|^2 + \int_0^T |Au^n|^2\mathrm{d}s + \int_0^T \theta(\|v^n\|)^2|B(u^n)|^2\mathrm{d}s + \int_0^T |F(u^n)|^2\mathrm{d}s\\
&=: |u_0|^2 + I_1 + I_2 + I_3, \qquad (9.12)\end{aligned}$$

I_1 is straightforward: $\mathbf{E}[I_1] \leq C$. Using the cutoff property for some $C > 0$ independent of n we have

$$\begin{aligned}\mathbf{E}[I_2] &\leq C\mathbf{E}\int_0^T \theta(\|v^n(s)\|)^2[|B(v^n(s))|^2 + |B(v^n(s),u_*^n(s))|^2]\mathrm{d}s\\
&\quad + C\mathbf{E}\int_0^T \theta(\|v^n(s)\|)^2[|B(u_*^n(s),v^n(s))|^2 + |B(u_*^n(s))|^2]\mathrm{d}s\\
&\leq C\mathbf{E}\int_0^T \theta(\|v^n(s)\|)^2[\|v^n(s)\|^2|Av^n(s)|^2 + \|u_*^n(s)\|^2|Au_*^n(s)|^2]\mathrm{d}s,\\
&\leq C\mathbf{E}\int_0^T [|Av^n(s)|^2 + \|u_*^n(s)\|^2|Au_*^n(s)|^2]\mathrm{d}s \leq C. \qquad (9.13)\end{aligned}$$

Similarly,

$$\mathbf{E}[I_3] \leq C\mathbf{E}\int_0^T (1+\|u^n(s)\|^2)\mathrm{d}s \leq C. \qquad (9.14)$$

Thus we have now shown that if $u_0 \in L^8(\Omega, \mathcal{F}_0, \mathbf{P}; V)$ then u^n is bounded in $L^4(\Omega; L^\infty(0,T;V)) \cap L^4(\Omega; L^2(0,T;D(A))) \cap L^2(\Omega; W^{\alpha,2}(0,T;H))$ independently of n. Hence, if for some u, $u^n \rightharpoonup u$ weakly/weak* in these spaces. We want to know if u càdlàg in time a.s. so that the integrals $\in_0^t G(u(s-))\mathrm{d}W(s)$ and $\int_{(0,T]}\int_{E_0} K(u(s-),\xi)\mathrm{d}\widehat{\pi}(s,\xi)$ are well-defined.
Next class: more a priori estimates to get u càdlàg .

10. Week 10

Theorem 10.1. *If $X_0 \subset\subset X \subset X_1$, where X_0, X, X_1 are Hilbert spaces then the injection $L^2(0,T;X_0) \cap W^{\alpha,2}(0,T;X_1) \subset L^2(0,T;X)$ is compact ($\alpha \in (0,1)$).*

In our example $X_0 = D(A), X = V, X_1 = H$.

10.1. Tightness.

Definition 10.2. *A set M of probability measures on ametric space (S,d) is tight if $\forall \varepsilon\ \exists K_\varepsilon \subset S$ compact s.t. $\mu(K_\varepsilon) < \varepsilon\ \forall \mu \in M$ (i.e. most of the mass is supported on a compact set K_ε).*

Note that K_ε does not depend on the individual measure $\mu \in M$.

Theorem 10.3. *(Prohorov's theorem) Let (S,d) be a complete separable metric space. Let M be a set of probability measures, then the following are equivalent.*

1. *$\bar{M}$ is compact in $P(S)$ (set of Borel probability measures on S.)*
2. *M is tight.*

Remark 10.1. *(1) implies M contains a weakly convergent sequence (though the limit may not be in M).*

Proposition 10.4. *If $u_0 \in L^8(\Omega, \mathcal{F}_0, \mathbf{P}, V)$ then the laws of approximation $\{u^n\}_{n=1}^\infty$ form a tight sequence of probability measures on $L^2(0,T;V)$.*

Proof. For all ε we want to find a compact set $K \subset L^2(0,T;V)$ s.t. $\mathbf{P}(u^n \notin K_\varepsilon) < \varepsilon$. Recall u^n is bounded in $L^4(\Omega; L^\infty(0,T;V)) \cap L^4(\Omega; L^2(0,T;D(A))) \cap L^2(\Omega; W^{\alpha,2}(0,T;H))$. Set $K_\varepsilon = \{v \in L^2(0,T;D(A)) \cap W^{\alpha,2}(0,T;H) : \|v\|_{L^2(D(A))} + \|v\|_{W^{\alpha,2}(0,T;H)} \leq r\}$ ($r(\varepsilon)$ to be determined later on). K_ε is bounded in the LHS space, so it is compact in $L^2(0,T;V)$.

$$\begin{aligned}
\mathbf{P}(u_\varepsilon \notin K_\varepsilon) &= \mathbf{P}(\|u^n\|_{L^2(0,T;D(A))} + \|u^n\|_{W^{\alpha,2}(0,T;H)} > r) \\
&\leq \mathbf{P}(\|u^n\|_{L^2(0,T;D(A))} > \frac{r}{2}) + \mathbf{P}(\|u^n\|_{W^{\alpha,2}(0,T;H)} > \frac{r}{2}) \\
(chebyshev) &\leq (\frac{2}{r})^4 \mathbf{E}[\|u^n\|^4_{L^2(0,T;D(A))}] + (\frac{2}{r})^2 \mathbf{E}[\|u^n\|^2_{W^{\alpha,2}(0,T;H)}] \\
&\leq C((\frac{2}{r})^4 + (\frac{2}{r})^2) \\
&\leq \varepsilon \quad \text{for large enough } r.
\end{aligned}$$

□

From the proposition we know the laws of $\{u^n\}$ has a weak convergent subsequence converging to some law μ.

Remark 10.2. *We do not know that the r.v. with this law is càdlàg or not.*

Remedy: Tightness on càdlàg space with "càdlàg " topology.

Definition 10.5. $D(0,T;H) := \{v : [0,T] \to H,\ v \text{ is càdlàg }\}$

Skorohod's convergence theorem: Let μ_n, $n \in \mathbb{N}$ of prob measures on separable space S s.t. $\mu_n \to \mu_\infty$ weakly. Then there exists r.v. X_n defined on some prob space $(\Omega, \mathcal{F}, \mathbf{P})$ s.t. law $X_n = \mu_n$ (including ∞) and $X_n \to X_\infty$, $\mathbf{P}$ a.s.
$D(0,T;H)$ is separable on Skorohod topology. If $v^n \in D(0,T;H)$, we hope $v^n \to v \in D(0,T;H)$. Difficulty: $L^\infty(0,T;H)$ norm only gives completeness but not separability.
Skorohod topology: $\Lambda = \{\lambda : [0,T] \to [0,T], \lambda \text{ is continuous, strictly increasing and onto}\}$. The distance $d(x,y) = \inf_{\lambda\in\Lambda}\{\sup|\log \frac{\lambda(t)-\lambda(s)}{t-s}| + \sup_{t\in[0,1]}|x(t)-x(\lambda(t))|\}$ makes $D(0,T;H)$ separable and complete.

Definition 10.6. *For $v^n, v \in D(0,T;H)$ we say that $v^n \to v$ in the Skorohod topology if there exists $\{\lambda_n\}_{n=1}^\infty \subset \Lambda$ s.t. (1) $v^n \circ \lambda_n \to v$ uniformly on $[0,T]$.*
(2) $\lambda_n \to id_{[0,T]}$ uniformly.

Remark 10.3. *We only need the following facts:*
(1) Uniformly convergent sequence in $D(0,T;H)$ converges in the Skorohod topology.
(2) $D(0,T;H)$ is separable under the Skorohod topology and metrizable by a complete metric.

Theorem 10.7. *Let $\{Y_n\}_{n=1}^\infty$ be a sequence of H-valued process defined on a filtered prob space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathbf{P})$ s.t. each Y_n is adapted and belongs to $D(0,T;H)$ a.s. Then for all $T > 0$, the following conditions (together) are sufficient to imply that the laws of $(Y_n)_{n=1}^\infty$ are tight on the càdlàg space $D(0,T;H)$ endowed with the Skorohod topology.*
(1) There exists a dense set $Q \subset [0,T]$ s.t. for all $t \in Q$, the laws of $(Y_n)_1^\infty$ are tight on H.
(2) Aldous condition: For every $\varepsilon > 0, \eta > 0$ there exists $\delta > 0$ s.t. for every sequence $(\tau_n)_1^\infty$ of stopping times bounded by T, we have

$$\sup_{n\geq 1}\sup_{t\in[0,\delta]} \mathbf{P}[|Y_n((\tau+t)\wedge T) - Y_n(\tau_n)|_H \geq \eta] \leq \varepsilon. \tag{10.1}$$

We apply this theorem to $\{u_n\}_1^\infty$.

Proposition 10.8. *The laws of $(u_n)_1^\infty$ are tight on $D(0,T;H)$ endowed with the Skorohod topology.*

Proof. (1) is usually easy to verify with $Q = [0,T]$. For a fixed $t \in [0,T]$ let $B_2 = \{v \in V, \|v\| \leq r\}$. Since $V \subset\subset H$, B_2 is bounded and closed in V, hence compact in H.

$$\begin{aligned}\mathbf{P}(u^n(t) \notin B_2) &= \mathbf{P}[\|u^n(t)\| > r] \\ &\leq \frac{1}{r^2}\mathbf{E}[\|u^n\|^2] \\ &\leq C/r^2 < \varepsilon \quad \text{for r sufficiently large.}\end{aligned}$$

(2) Verify Aldous condition: Let $\{\tau_n\}_1^\infty$ be stopping times bounded by T. By Chebyshev it suffices to verify that

$$\sup_{n\geq 1}\sup_{t\in[0,\delta]} (1/\eta)^2 \mathbf{E}[|u_n((t+\tau_n)\wedge T) - u_n(\tau_n)|_H^2] \to 0 \quad \text{as } \delta \to 0.$$

Then by Cauchy-Schwarz we obtain,

$$
\begin{aligned}
&|u^n((\tau_n+t)\wedge T)-u^n(\tau^n)|^2 \lesssim\ t\int_{\tau_n}^{(\tau_n+t)\wedge T}|Au^n(s)|^2\mathrm{d}s + t\int_{\tau_n}^{(\tau_n+t)\wedge T}(\theta(\|v^n(s)\|))^2|B(u^n(s))|^2\mathrm{d}s\\
&+t\int_{\tau_n}^{(\tau_n+t)\wedge T}|F(u^n(s))|^2\mathrm{d}s + \Big|\int_{\tau_n}^{(\tau_n+t)\wedge T}P_nG(u^n(s-))\mathrm{d}W(s)\Big|^2\\
&+\Big|\int_{(\tau_n,(\tau_n+t)\wedge T]}\int_{E_0}P_nK(u^n(s-),\xi)\mathrm{d}\widehat{\pi}(s,\xi)\Big|^2\\
&=: J_1+J_2+J_3+J_4+J_5.
\end{aligned}
$$

Now the expectation $\mathbf{E}$ of the first 3 terms is bounded by $\delta\mathbf{E}\int_0^T ..\mathrm{d}s \to 0$ as $\delta\to 0$ thanks to Lemma 1 and Proposition 1 from last class. The next two terms: Using Itô's isometry we have:

$$
\begin{aligned}
\mathbf{E}\Big|\int_{\tau_n}^{(\tau_n+t)\wedge T}P_nG(u^n(s-))\mathrm{d}W(s)\Big|^2 &= \mathbf{E}\Big[\int_{\tau_n}^{(\tau_n+t)\wedge T}\|P_nG(u^n(s-))\|^2_{L_2(U_0,H)}\,\mathrm{d}s\Big]\\
&\le C\mathbf{E}\Big[\int_{\tau_n}^{(\tau_n+t)\wedge T}(1+|u^n(s)|^2)\mathrm{d}s\Big]\\
&\le Ct\mathbf{E}\Big[\Big(1+\sup_{t\in[0,T]}|u^n(s)|^2\Big)\Big]\\
&\le C\delta\to 0\quad \text{as } \delta\to 0.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathbf{E}\Big|\int_{(\tau_n,(\tau_n+t)\wedge T]}\int_{E_0}P_nK(u^n(s-),\xi)\mathrm{d}\widehat{\pi}(s,\xi)\Big|^2 &= \mathbf{E}\Big[\int_{\tau_n}^{(\tau_n+t)\wedge T}\int_{E_0}|P_nK(u^n(s-),\xi)|^2\mathrm{d}\nu(\xi)\mathrm{d}s\Big]\\
&\le C\mathbf{E}\Big[\int_{\tau_n}^{(\tau_n+t)\wedge T}(1+|u^n(s)|^2)\mathrm{d}s\Big]\\
&\le Ct\mathbf{E}\Big[\Big(1+\sup_{s\in[0,T]}|u^n(s)|^2\Big)\Big]\\
&\le C\delta\to 0\quad \text{as } \delta\to 0.
\end{aligned}
$$

This shows that the Aldous condition is satisfied and the laws of $(u^n)_{n=1}^\infty$ are tight in $\mathcal{D}([0,T];H)$ endowed with the Skorohod topology. □

Using a standard argument and combining the above two propositions we obtain that the laws of $\{u_n\}_1^\infty$ are tight in the space $\mathcal{D}(0,T;H)\cap L^2([0,T];V) := \mathcal{X}$.

Next: Convergence lemmas:

Let $\mathcal{N}^{\#*}_{[0,\infty)\times E}$ denote the space of counting measures on $[0,\infty)\times E$ that are finite on bounded sets and let $\Upsilon := H\times\mathcal{X}\times C([0,T];U)\times\mathcal{N}^{\#*}_{[0,\infty)\times E}$. Fact: the spaces $\mathcal{N}^{\#*}_{[0,\infty)\times E}$ and $C([0,T];U)$ (cf. Daley, Vere Jones book– an introduction to point processes) are separable and metrizable by a complete metric. Thus we see that the sequence of probability measures, $\mu^n_W(\cdot) := \mathbf{P}(W\in\cdot)$, being constantly equal to one element, is tight on $C([0,T];U)$. Similarly we find out that the law of π is tight on $\mathcal{N}^{\#*}_{[0,\infty)\times E}$. It is also clear that $u_0^n\to u_0$ $\mathbf{P}$-a.s. in H, that is the laws of $(u_0^n)_{n=1}^\infty$ are tight on H and that their weak limit is the measure $\mu_0 = \mathbf{P}(u_0\in\cdot)$.

Thus the sequence of the joint probability measures μ^n of the approximating sequence $(u_0^n, u^n, W, \pi)_{n=1}^\infty$ is tight on the space Υ. The Prohorov theorem implies $(\mu^n)_{n=1}^\infty$ is weakly compact over Υ. This implies that there exists a probability measure μ^∞ on Υ such that for some subsequence μ^n converges weakly to μ^∞.

We will use the Skorohod representation theorem to upgrade the weak convergence to a.s. convergence in Υ for random variables defined on a new probability space.

Proposition 10.9. *Suppose that u_0 has law μ_0. Then there exists a filtered probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathbf{P}})$ and Υ-valued random variables $(\widetilde{u}_0, \widetilde{u}, \widetilde{W}, \widetilde{\pi})$ and $\Big((\widetilde{u}_0^n, \widetilde{u}^n, \widetilde{W}_n, \widetilde{\pi}_n)\Big)_{n\in\Lambda}$ such that*

- *i) $(\widetilde{u}_0^n, \widetilde{u}^n, \widetilde{W}_n, \widetilde{\pi}_n) \overset{\mathcal{D}}{=} (u_0^n, u^n, W, \pi)$ for every $n \in \Lambda$, $(\widetilde{u}_0, \widetilde{u}, \widetilde{W}, \widetilde{\pi})$ has law μ^∞, $\widetilde{u}_0$ has law μ_0 and $(\widetilde{u}_0^n, \widetilde{u}^n, \widetilde{W}_n, \widetilde{\pi}_n) \to (\widetilde{u}_0, \widetilde{u}, \widetilde{W}, \widetilde{\pi})$ in Υ, $\widetilde{\mathbf{P}}$-a.s., as $n \to \infty$ along a subsequence,*
- *ii) $(\widetilde{W}_n, \widetilde{\pi}_n) = (\widetilde{W}, \widetilde{\pi})$ everywhere on $\widetilde{\Omega}$ and*
- *iii) $\widetilde{u}^n(0) = \widetilde{u}_0^n$ $\widetilde{\mathbf{P}}$-a.s., i.e., $\widetilde{u}^n$ is an $\widetilde{\mathcal{F}}_t$-adapted process with càdlàg paths in H, $\widetilde{\mathbf{P}}$-a.s., $\widetilde{u}_0^n$ is $\widetilde{\mathcal{F}}_0$-measurable and*

$$\begin{aligned}\widetilde{u}^n(t) + \int_0^t A\widetilde{u}^n(s)\mathrm{d}s + \int_0^t \theta(\|\widetilde{u}^n(s) - \widetilde{u}_*^n(s)\|)P_n B(\widetilde{u}^n(s))\mathrm{d}s + \int_0^t P_n F(\widetilde{u}^n(s))\mathrm{d}s \\ = \widetilde{u}_0^n + \int_0^t P_n G(\widetilde{u}^n(s))\mathrm{d}\widetilde{W}(s) + \int_{(0,t]}\int_{E_0} P_n K(\widetilde{u}^n(s-), \xi)\mathrm{d}\widehat{\widetilde{\pi}}(s,\xi),\end{aligned} \tag{10.2}$$

holds in H_n, $\widetilde{\mathbf{P}}$-a.s., for all $t \in [0,T]$, where $\widetilde{u}_*^n$ solves the random ODE

$$\begin{cases}\mathrm{d}\widetilde{u}_*^n + A\widetilde{u}_*^n\mathrm{d}t = 0, \\ \widetilde{u}_*^n(0) = \widetilde{u}_0^n.\end{cases} \tag{10.3}$$

Now we will pass to the limit $n \to \infty$ in (10.2). First recall Vitali's convergence lemma:

Lemma 10.10. *Let $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}})$ be a probability space, let X be a Banach space and let $p \in [1, \infty)$. Let $f, f_1, f_2, \ldots \in L^p(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}}; X)$ and suppose that*

- *i) $|f_n - f|_X \to 0$ in probability as $n \to \infty$ and*
- *ii) $\sup_{n\geq 1} \widetilde{\mathbf{E}}|f_n|_X^q < \infty$ for some $q \in (p, \infty)$.*

Then $f_n \to f$ in the space $L^p(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}}; X)$.

Now we will state one of the two main theorems:

Theorem 10.11. *Suppose that $\mathbf{E}\|u_0\|^8 < \infty$. Then $\widetilde{u}$ is a global martingale solution to the cutoff-SPDE with respect to the stochastic basis $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathbf{P}}, \widetilde{W}, \widetilde{\pi})$ with initial condition $\widetilde{u}_0$, where $\widetilde{u}_0$ has the same law as that of u_0.*

Proof.

Lemma 10.12. *Suppose that $\mathbf{E}\|u_0\|^8 < \infty$. Then $\widetilde{u} \in L^4(\widetilde{\Omega}; L^2([0,T]; D(A))) \cap L^4(\widetilde{\Omega}; L^\infty([0,T]; V))$,*

$$\widetilde{\mathbf{E}}\int_0^T \|\widetilde{u}(s)\|^2 |A\widetilde{u}(s)|^2 \mathrm{d}s < \infty \tag{10.4}$$

and the following statements hold along a subsequence:

- *i) $\widetilde{u}^n \to \widetilde{u}$ in $L^1(\widetilde{\Omega}; L^1([0,T]; V))$ as $n \to \infty$.*
- *ii) $\widetilde{u}^n \rightharpoonup \widetilde{u}$ weakly in the space $L^2(\widetilde{\Omega}; L^2([0,T]; V))$ as $n \to \infty$.*

iii) $\widetilde{u}^n \rightharpoonup \widetilde{u}$ *weakly in the space* $L^4(\widetilde{\Omega}; L^2([0,T]; D(A)))$ *as* $n \to \infty$.
iv) $\widetilde{u}^n \to \widetilde{u}$ *weak** *in the space* $L^4(\widetilde{\Omega}; L^\infty([0,T]; V))$ *as* $n \to \infty$.

Proof. (i) Proposition 10.9 implies $\widetilde{u}^n \to \widetilde{u}$, $\widetilde{\mathbf{P}}$-a.s., in $L^2([0,T];V)$ and thus in $L^1([0,T];V)$. We know that the sequence $(\widetilde{u}^n)_{n\in\Lambda}$ is bounded in $L^2(\widetilde{\Omega}; L^1([0,T];V))$. Thus Vitali's convergence lemma implies that the convergence holds in $L^1(\widetilde{\Omega}; L^1([0,T];V))$.
(ii) Cauchy-Schwarz inequality and part i) imply that for $\phi \in V$ and a set $A \in \widetilde{\mathcal{F}} \otimes \mathcal{B}([0,T])$

$$\widetilde{\mathbf{E}} \int_0^T |((\widetilde{u}^n(s) - \widetilde{u}(s), \phi\chi_\Gamma))| \mathrm{d}s \to 0 \qquad \text{as } n \to \infty,\ n \in \Lambda.$$

The rest follow from the Banach-Alaoglu theorem. □

Convergence of noise terms:

Lemma 10.13. $P_n G(\widetilde{u}^n) \to G(\widetilde{u})$ $\widetilde{\mathbf{P}}$*-a.s., in the space* $L^2([0,T]; L_2(U_0, V))$ *as* $n \to \infty$.

Corollary 2. *The processes* $\left((\int_0^t P_n G(\widetilde{u}^n(s-)) \mathrm{d}\widetilde{W}(s))_{t\in[0,T]} \right)_{n\in\Lambda}$ *converge in the space* $L^1(\widetilde{\Omega}; L^1([0,T];V))$ *to* $(\int_0^t G(\widetilde{u}(s-)) \mathrm{d}\widetilde{W}(s))_{t\in[0,T]}$.

Corollary 2 follows from Proposition 4.16 in Da Prato and by applying Lemma 10.10. In order to apply this lemma, we are also required to verify that $\left((\int_0^t P_n G(\widetilde{u}^n(s)) \mathrm{d}\widetilde{W}(s))_{t\in[0,T]} \right)_{n\in\Lambda}$ is a sequence bounded independently of n in the space $L^2(\widetilde{\Omega}; L^1([0,T];V))$, but this follows easily from the growth conditions.

Similarly one can show,

Lemma 10.14. $P_n K(\widetilde{u}^n(s-), \xi) \to \widetilde{K}$ *as* $n \to \infty$, $\widetilde{\mathbf{P}}$*-a.s., in the space* $L^2([0,T] \times E_0, \mathrm{d}t \otimes \mathrm{d}\nu; V)$.

Corollary 3. *The processes* $\left((\int_{(0,t]} \int_{E_0} P_n K(\widetilde{u}^n(s-), \xi) \mathrm{d}\widehat{\widetilde{\pi}}(s,\xi))_{t\in[0,T]} \right)_{n\in\Lambda}$ *converge in the space* $L^1(\widetilde{\Omega}; L^1([0,T];V))$ *to* $(\int_{(0,t]} \int_{E_0} K(\widetilde{u}(s-), \xi) \mathrm{d}\widehat{\widetilde{\pi}}(s,\xi))_{t\in[0,T]}$.

Thus for $\phi \in D(A)$ and a set $\Gamma \in \widetilde{\mathcal{F}} \otimes \mathcal{B}([0,T])$

$$\widetilde{\mathbf{E}} \int_0^T |\chi_\Gamma(\phi, \widetilde{u}^n(t) - \widetilde{u}(t))| \mathrm{d}t \to 0,$$

as $n \to \infty$ along Λ.

By Corollary 2 we have $\widetilde{\mathbf{E}} \int_0^T \left| \chi_\Gamma(\phi, \int_0^t [P_n G(\widetilde{u}^n(s-)) - G(\widetilde{u}(s-))] \mathrm{d}\widetilde{W}(s)) \right| \mathrm{d}t \to 0$.
By Corollary 3 we have $\widetilde{\mathbf{E}} \int_0^T \left| \chi_\Gamma(\phi, \int_{(0,t]} \int_{E_0} [P_n K(\widetilde{u}^n(s-), \xi) - K(\widetilde{u}(s-), \xi)] \mathrm{d}\widehat{\widetilde{\pi}}(s,\xi)) \right| \mathrm{d}t \to 0$.
For the linear term we have

$$\begin{aligned} \widetilde{\mathbf{E}} \int_0^T \Big| \int_0^t (A\widetilde{u}^n(s) - A\widetilde{u}(s), \phi\chi_\Gamma) \mathrm{d}s \Big| \mathrm{d}t &= \widetilde{\mathbf{E}} \int_0^T \chi_\Gamma \Big| \int_0^t ((\widetilde{u}^n(s) - \widetilde{u}(s), \phi)) \mathrm{d}s \Big| \mathrm{d}t \\ &\leq T\|\phi\| \widetilde{\mathbf{E}} \int_0^T \|\widetilde{u}^n(s) - \widetilde{u}(s)\| \mathrm{d}s. \end{aligned} \tag{10.5}$$

11. Week 11

For the B term: By the triangle inequality we have

$$
\begin{aligned}
\alpha_n(t) &:= \Big| \int_0^t \chi_\Gamma(\phi, \theta(\|\widetilde{v}^n(s)\|) P_n B(\widetilde{u}^n(s)) - \theta(\|\widetilde{v}(s)\|) B(\widetilde{u}(s)))\mathrm{d}s \Big| \\
&\le \Big| \int_0^t \theta(\|\widetilde{v}^n(s)\|) \chi_\Gamma(\phi, P_n B(\widetilde{u}^n(s)) - P_n B(\widetilde{u}(s)))\mathrm{d}s \Big| \\
&\quad + \Big| \int_0^t \theta(\|\widetilde{v}^n(s)\|) \chi_\Gamma((I-P_n)\phi, B(\widetilde{u}(s)))\mathrm{d}s \Big| \\
&\quad + \Big| \int_0^t [\theta(\|\widetilde{v}^n(s)\|) - \theta(\|\widetilde{v}(s)\|)] \chi_\Gamma(\phi, P_n B(\widetilde{u}(s)))\mathrm{d}s \Big| \\
&=: \alpha_{n,1}(t) + \alpha_{n,2}(t) + \alpha_{n,3}(t).
\end{aligned}
$$

First we treat $\alpha_{n,1}$:

$$
\begin{aligned}
\alpha_{n,1}(t) &= \Big| \int_0^t \theta(\|\widetilde{u}^n(s)\|) \chi_\Gamma(\phi, P_n B(\widetilde{u}^n(s) - \widetilde{u}(s), \widetilde{u}(s)) + P_n B(\widetilde{u}^n(s), \widetilde{u}^n(s) - \widetilde{u}(s)))\mathrm{d}s \Big| \\
&\le \int_0^t \theta(\|\widetilde{v}^n(s)\|) |(\widetilde{u}(s), B(\widetilde{u}^n(s) - \widetilde{u}(s), P_n\phi)) + (\widetilde{u}^n(s) - \widetilde{u}(s), B(\widetilde{u}^n(s), P_n\phi))| \mathrm{d}s \\
&\le C \int_0^T |AP_n\phi| \theta(\|\widetilde{v}^n(s)\|) \|\widetilde{u}^n(s) - \widetilde{u}(s)\| (\|\widetilde{u}(s)\| + \|\widetilde{u}^n(s)\|)\, \mathrm{d}s \\
&\le C|A\phi| \int_0^T \theta(\|\widetilde{v}^n(s)\|) \|\widetilde{u}^n(s) - \widetilde{u}(s)\| (\|\widetilde{u}(s)\| + \|\widetilde{v}^n(s)\| + \|\widetilde{u}_*^n(s)\|)\, \mathrm{d}s. \qquad (11.1)
\end{aligned}
$$

Cauchy-Schwarz inequality we obtain

$$
\begin{aligned}
\alpha_{n,1}(t) &\le C|A\phi| \|\widetilde{u}^n - \widetilde{u}\|_{L^2([0,T];V)} \Big(\int_0^T (\|\widetilde{u}(s)\|^2 + R^2 + \|\widetilde{u}_*^n(s)\|^2)\mathrm{d}s \Big)^{1/2} \\
&\le C|A\phi| \|\widetilde{u}^n - \widetilde{u}\|_{L^2([0,T];V)} \cdot \left(R + \sup_{s\in[0,T]} [\|\widetilde{u}(s)\| + \|\widetilde{u}_*^n(s)\|] \right).
\end{aligned}
$$

the triangle inequality and the cutoff property gives

$$
\begin{aligned}
\alpha_{n,1}(t) &\le C|A\phi| \int_0^T \left(\|\widetilde{u}(s)\|^2 + R^2 + \|\widetilde{u}_*^n(s)\|^2 \right) \mathrm{d}s \\
&\le TC|A\phi| \left(R^2 + \sup_{s\in[0,T]} [\|\widetilde{u}(s)\|^2 + \|\widetilde{u}_*^n(s)\|^2] \right) \\
&\le TC|A\phi| \left(R^2 + \sup_{s\in[0,T]} [\|\widetilde{u}(s)\|^2] + \|\widetilde{u}_0\|^2 \right). \qquad (11.2)
\end{aligned}
$$

The right-hand side of inequality (11.2) (which does not depend on t) belongs to $L^1(\widetilde{\Omega} \times [0,T])$. Therefore, the dominated convergence theorem implies that $\alpha_{n,1} \to 0$ in $L^1(\widetilde{\Omega} \times [0,T])$ as $n \to \infty$. Next,

$$
\begin{aligned}
\alpha_{n,2}(t) &= \Big| \int_0^t \theta(\|\widetilde{v}^n(s)\|) \chi_\Gamma(\widetilde{u}(s), B(\widetilde{u}(s), (I-P_n)\phi))\mathrm{d}s \Big| \\
&\le C \int_0^T |A(I-P_n)\phi| \cdot \|\widetilde{u}(s)\|^2 \mathrm{d}s \le CT|(I-P_n)A\phi| \cdot \sup_{s\in[0,T]} \|\widetilde{u}(s)\|^2. \qquad (11.3)
\end{aligned}
$$

Since $|(I - P_n)A\phi| \to 0$, the right-hand side tends to 0, $d\widetilde{\mathbf{P}} \otimes dt$-a.e. Furthermore, since $|(I - P_n)A\phi| \leq |A\phi|$ we see that the right hand side of (11.3) is dominated by a function in $L^1(\widetilde{\Omega} \times [0, T])$. It follows from the dominated convergence theorem that $\alpha_{n,2} \to 0$ in $L^1(\widetilde{\Omega} \times [0, T])$ as $n \to \infty$.

$$\begin{aligned} \alpha_{n,3}(t) &= \Big| \int_0^t [\theta(\|\widetilde{v}^n(s)\|) - \theta(\|\widetilde{v}(s)\|)] \chi_\Gamma(\widetilde{u}(s), B(\widetilde{u}(s), P_n\phi)) ds \Big| \\ &\leq C|AP_n\phi| \int_0^T |\theta(\|\widetilde{v}^n(s)\|) - \theta(\|\widetilde{v}(s)\|)| \cdot \|\widetilde{u}(s)\|^2 ds \end{aligned} \tag{11.4}$$

the integrand on the right-hand side of (11.4) is dominated by the integrable function $2C|A\phi|\|\widetilde{u}(s)\|^2$ and tends to 0, $d\widetilde{\mathbf{P}} \otimes dt$-a.e., as $n \to \infty$ because $\widetilde{v}^n \to \widetilde{v}$ in V, $d\widetilde{\mathbf{P}} \otimes dt$-a.e., as $n \to \infty$ along Λ' and because the cutoff function θ is continuous. This shows that $\alpha_n \to 0$ in $L^1(\widetilde{\Omega} \times [0, T])$ as $n \to \infty$ along Λ'.

Next: collect results and finish the proof. Apply $\chi_\Gamma(\phi, \cdot)$ and let $n \to \infty$ in $L^1(\widetilde{\Omega} \otimes [0, T])$,

$$\begin{aligned} &\chi_\Gamma(\phi, \widetilde{u}(t)) + \int_0^t \chi_\Gamma(A\widetilde{u}(s), \phi) ds + \int_0^t \theta(\|\widetilde{v}(s)\|) \chi_\Gamma(\phi, B(\widetilde{u}(s))) ds + \int_0^t \chi_\Gamma(F(\widetilde{u}(s)), \phi) ds \\ &\quad = \chi_\Gamma(\phi, \widetilde{u}_0) + \int_0^t \chi_\Gamma(\phi, G(\widetilde{u}(s-)) d\widetilde{W}(s)) + \int_{(0,t]} \int_{E_0} \chi_\Gamma(\phi, K(\widetilde{u}(s-), \xi)) d\widehat{\widetilde{\pi}}(s, \xi), \end{aligned}$$

$d\widetilde{\mathbf{P}} \otimes dt$-a.e. Since $D(A)$ is separable and dense in H, it follows that the equality above holds $d\widetilde{\mathbf{P}} \otimes dt$-a.e. for all $\phi \in H$. This means that

$$\begin{aligned} &\chi_\Gamma \widetilde{u}(t) + \int_0^t \chi_\Gamma A\widetilde{u}(s) ds + \int_0^t \theta(\|\widetilde{v}(s)\|) \chi_\Gamma B(\widetilde{u}(s)) ds + \int_0^t \chi_\Gamma F(\widetilde{u}(s)) ds \\ &\quad = \chi_\Gamma \widetilde{u}_0 + \int_0^t \chi_\Gamma G(\widetilde{u}(s-)) d\widetilde{W}(s) + \int_{(0,t]} \int_{E_0} \chi_\Gamma K(\widetilde{u}(s-), \xi) d\widehat{\widetilde{\pi}}(s, \xi) \end{aligned} \tag{11.5}$$

in the space H, $d\widetilde{\mathbf{P}} \otimes dt$-a.e. □

11.1. **Pathwise uniqueness.** Next we apply the well known result by Gyöngy and Krylov that extends the Yamada–Watanabe theorem to infinite dimension. This theorem states that the existence of martingale solutions and the pathwise uniqueness imply the existence of a pathwise solution to the truncated system.

So we will next prove pathwise uniqueness.

Definition 11.1. *We say that* ***pathwise uniqueness*** *holds for the truncated equation if for every pair of global martingale solutions u and v with respect to the same stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P}, W, \pi)$ one has*

$$\mathbf{P}[\mathbf{1}_{\{u(0)=v(0)\}}(u(t) - v(t)) = 0 \ \forall t \in [0, T]] = 1.$$

Let u and u' be two solutions to the cutoff equation up to a stopping time τ such that $u, u' \in L^4(\Omega; L^\infty([0, T]; V)) \cap L^4(\Omega; L^2([0, T]; D(A)))$. For each positive integer n, we define the random variable

$$\tau_n := T \wedge \inf\{t \in [0, \tau) : \textstyle\int_0^t [\|u(s)\|^2 \cdot |Au(s)|^2 + \|u'(s)\|^2 \cdot |Au'(s)|^2] ds \geq n\}, \tag{11.6}$$

Since the integral in the definition is adapted and continuous in t a.s. it follows that τ_n is an $\mathcal{F}_t$-stopping time.

Proposition 11.2. *Pathwise uniqueness holds for the truncated equation. Moreover, if* u *and* u' *are local pathwise solutions to the truncated SPDE up to the same stopping time* τ *relative to the same stochastic basis* $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0}, \mathbf{P}, W, \pi)$ *with respective initial conditions* u_0 *and* u_0'*, then we have*

$$\mathbf{P}[\mathbf{1}_{\{u_0=u_0'\}}(u(t)-u'(t))=0\ \forall t\in[0,\tau)]=1.$$

Proof. Let $w(t) := (u(t)-u'(t))$ and assume $u_0 = u_0'$. We will show that

$$\mathbf{E}\left(\sup_{t\in[0,\tau)}\|w(t)\|^2\right)=0. \tag{11.7}$$

Let u_* and u_*' solve the ODE. Define $v := u - u_*$ and $v' := u' - u_*$. We have the following SDE for w:

$$\begin{aligned}
&\mathrm{d}w + [Aw + \theta(\|v\|)B(u) - \theta(\|v'\|)B(u') + F(u) - F(u')]\mathrm{d}t \\
&\qquad = (G(u(t-)) - G(u'(t-)))\mathrm{d}W(t) + \int_{E_0}(K(u(t-),\xi) - K(u'(t-),\xi))\mathrm{d}\widehat{\pi}(t,\xi), \\
&w(0) = 0.
\end{aligned}$$

We apply Ito's formula:

$$\begin{aligned}
\|w(t)\|^2 + 2\int_0^t |Aw(s)|^2\mathrm{d}s &= 2\int_0^t(\theta(\|v'(s)\|)B(u'(s)) - \theta(\|v(s)\|)B(u(s)), Aw(s))\mathrm{d}s \\
&+ 2\int_0^t (F(u'(s)) - F(u(s)), Aw(s))\mathrm{d}s + 2\int_0^t((w(s),[G(u(s-)) - G(u'(s-))]\mathrm{d}W(s))) \\
&+ \int_0^t \|G(u(s)) - G(u'(s-))\|^2_{L_2(U_0,V)}\,\mathrm{d}s + 2\int_{(0,t]}\int_{E_0}((w(s-),K(u(s-),\xi) - K(u'(s-),\xi)))\mathrm{d}\widehat{\pi}(s,\xi) \\
&+ \int_{(0,t]}\int_{E_0}\|K(u(s-),\xi) - K(u'(s-),\xi)\|^2\mathrm{d}\pi(s,\xi) \\
&=: I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t) + I_6(t),
\end{aligned}$$

a.s. for all $t\in[0,\tau)$.

$$\begin{aligned}
\mathbf{E}\left[\sup_{s\in[0,t\wedge\tau_n]}|I_1(s)|\right] &\leq \mathbf{E}\int_0^{t\wedge\tau_n}|\theta(\|v(s)\|) - \theta(\|v'(s)\|)|\cdot|(B(u(s)),Aw(s))|\mathrm{d}s \\
&\quad + \mathbf{E}\int_0^{t\wedge\tau_n}|\theta(\|v'(s)\|)|\cdot|(B(u(s)) - B(u'(s)),Aw(s))|\mathrm{d}s \\
&=: J_1 + J_2.
\end{aligned} \tag{11.8}$$

We estimate J_1 using the fact that the cutoff function θ is Lipschitz to obtain

$$\begin{aligned}
J_1 &\leq C\mathbf{E}\int_0^{t\wedge\tau_n}\|w(s)\|\cdot\|u(s)\|\cdot|Au(s)|\cdot|Aw(s)|\mathrm{d}s \\
&\leq \frac{1}{2}\mathbf{E}\int_0^{t\wedge\tau_n}|Aw(s)|^2\mathrm{d}s + C\int_0^{t\wedge\tau_n}\|w(s)\|^2\|u(s)\|^2|Au(s)|^2\mathrm{d}s.
\end{aligned}$$

To estimate J_2 we use the bilinearity of B and Young's inequality to obtain

$$J_2 \leq \mathbf{E}\int_0^{t\wedge\tau_n}[|(B(w(s),u(s)),Aw(s))| + |(B(u'(s),w(s)),Aw(s))|]\mathrm{d}s$$

$$\leq C\mathbf{E}\int_0^{t\wedge\tau_n}|Aw(s)|^{3/2}\|w(s)\|^{1/2}\cdot\Big[\|u(s)\|^{1/2}|Au(s)|^{1/2}+\|u'(s)\|^{1/2}|Au'(s)|^{1/2}\Big]\mathrm{d}s.$$

$$\leq\frac{1}{2}\mathbf{E}\int_0^{t\wedge\tau_n}|Aw(s)|^2\mathrm{d}s+C\int_0^{t\wedge\tau_n}\|w(s)\|^2\Big[\|u(s)\|^2|Au(s)|^2+\|u'(s)\|^2|Au'(s)|^2\Big]\mathrm{d}s. \tag{11.9}$$

Using the Lipschitz assumption, we easily obtain

$$\mathbf{E}\left[\sup_{s\in[0,t\wedge\tau_n]}[|I_4(s)|+|I_6(s)|]\right]\leq C\mathbf{E}\int_0^{t\wedge\tau_n}\|w(s)\|^2\mathrm{d}s. \tag{11.10}$$

Using the BDG inequalities (4.6) and (5.12), Young's inequality and the Lipschitz condition we obtain

$$\mathbf{E}\left[\sup_{s\in[0,t\wedge\tau_n]}[|I_3(s)|+|I_5(s)|]\right]\leq\frac{1}{2}\mathbf{E}\left[\sup_{s\in[0,t\wedge\tau_n]}\|w(s)\|^2\right]+\mathbf{E}\int_0^{t\wedge\tau_n}\|w(s)\|^2\mathrm{d}s. \tag{11.11}$$

Collecting all the terms we find that

$$\mathbf{E}\Big[\sup_{s\in[0,t\wedge\tau_n]}\|w(s)\|^2\Big]\leq C\mathbf{E}\int_0^{t\wedge\tau_n}\|w(s)\|^2\Big[1+\|u(s)\|^2|Au(s)|^2+\|u'(s)\|^2|Au'(s)|^2\Big]\mathrm{d}s. \tag{11.12}$$

Observe that the process $\phi(s):=1+\|u(s)\|^2|Au(s)|^2+\|u'(s)\|^2|Au'(s)|^2$ satisfies

$$\int_0^{\tau_n\wedge\tau}\phi(s)\mathrm{d}s\leq n,\qquad\mathbf{P}\text{-a.s..}$$

As a consequence, we have

$$\begin{aligned}\mathbf{E}\int_0^{\tau_n\wedge\tau}\|w(s)\|^2\phi(s)\mathrm{d}s&\leq\mathbf{E}\Big[\left(\sup_{s\in[0,\tau_n\wedge\tau]}\|w(s)\|^2\right)\int_0^{\tau_n\wedge\tau}\phi(s)\mathrm{d}s\Big]\\&\leq n\mathbf{E}\left(\sup_{s\in[0,\tau_n\wedge\tau]}\|w(s)\|^2\right)<\infty,\end{aligned}$$

because $w\in L^2(\Omega;L^\infty([0,T];V))$. We may now apply the stochastic Gronwall inequality, which completes the proof. □

We now return to main PDE by removing the cutoff function θ from equation. Let $u_0\colon\Omega\to V$ be $\mathcal{F}_0$-measurable and let (u,σ) be the maximal local pathwise solution to the truncated equations with initial condition u_0. It is clear that u is a local solution to the main SPDE up to the stopping time

$$\tau:=\sigma\wedge\inf\{t>0:\|u(t)-u_*(t)\|>R/2\}. \tag{11.13}$$

Since the difference $u-u_*$ starts at 0 and is right-continuous in the V-norm at time $t=0$, $\mathbf{P}$-a.s., it follows that $\tau>0$, $\mathbf{P}$-a.s. To sum up:

Theorem 11.3. *Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq0},\mathbf{P},W,\pi)$ be a stochastic basis and let $u_0\colon\Omega\to V$ be an arbitrary $\mathcal{F}_0$-measurable random variable. Then there exists an $\mathcal{F}_t$-adapted, $D(A)$-valued process $(u(t))_{t\in[0,T]}$ and an $\mathcal{F}_t$-stopping time τ such that $\tau>0$ a.s., u is càdlàg in the V-norm on $[0,\tau)$ a.s., the pair (u,τ) is a local solution to the main SPDE with $\mathcal{K}=0$ i.e. it satisfies for all $t\in[0,\tau)$*

(11.14)

$$u(t)+\int_0^t[Au(s)+B(u(s))+F(u(s))]\mathrm{d}s=u_0+\int_0^tG(u(s-))\mathrm{d}W(s)+\int_{(0,t]}\int_{E_0}K(u(s-),\xi)\mathrm{d}\widehat{\pi}(s,\xi)$$

Example 11.4. Stochastic Navier-Stokes: For $n = 2$

$$\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = G(u)\mathrm{d}W + \int_{E_0} K(u,\xi)\mathrm{d}\widehat{\pi} & \text{in } (0,T)\times\mathcal{O} \\ \text{div } u = 0 & \text{on } (0,T)\times\partial\mathcal{O} \\ u = 0 & \text{on } (0,T)\times\partial\mathcal{O} \\ u(0) = u_0 & \text{on } \{0\}\times\mathcal{O}, \end{cases} \tag{11.15}$$

where we look for the vector $u = (u_1, u_2)$ and $p \in \mathbb{R}$. Here $A = -\Delta$, $B = (u\cdot\nabla)u = (..., \sum_{i=1}^2 (u_i\partial_i u_j), ..)$ and $F = 0$.

Example 11.5 (Chafee-Infante Equation)**.**

$$\begin{cases} \partial_t u - \Delta u + \lambda(u^3 - u) = G(u)\mathrm{d}W + \int_{E_0} K(u,\xi)\mathrm{d}\widehat{\pi} & \text{in } (0,T)\times\mathcal{O} \\ u = 0 & \text{on } (0,T)\times\partial\mathcal{O} \\ u(0) = u_0 & \text{on } \{0\}\times\mathcal{O}, \end{cases} \tag{11.16}$$

where $\mathcal{O}$ is an open, bounded subset of $\mathbb{R}^d$, $d \le 3$, with smooth boundary $\partial\mathcal{O}$. In this example we can in fact show that $\mathbf{P}[\tau < T] = 0$. That is the solution is global.

Piecing out argument: Want to show the existence of solution to

(11.17)

$$u(t) + \int_0^t [Au(s) + B(u(s)) + F(u(s))]\mathrm{d}s = u_0 + \int_0^t G(u(s-))\mathrm{d}W(s) + \int_{(0,t]}\int_{E_0} K(u(s-),\xi)\mathrm{d}\widehat{\pi}(s,\xi) + \int_{(0,t]}\int_{E\setminus E_0} \mathcal{K}(u(s-),\xi)\mathrm{d}\pi(s,\xi)$$

Recall

$$\int_{(0,t]}\int_{E\setminus E_0} \mathcal{K}(u(s-),\xi)\mathrm{d}\pi(s,\xi) = \sum_{s\in(0,t]} \mathcal{K}(u(s-),\Delta L(s))\chi_{E\setminus E_0}(\Delta L(s))$$

Let us enumerate the elements of $\{s \in (0,t] : \Delta L(s) \in E\setminus E_0\}$ as $\sigma_1 < \sigma_2 < \sigma_3 < \cdots$. The method in the piecing-out argument is to construct a solution to (11.17) by solving equation (11.14) up to time $\sigma_1 \wedge \tau$, where τ is the time of existence of the local solution to (11.14). If $\tau \le \sigma_1$, then the solution to (11.14) blows up at or before the jump at time σ_1, so the construction stops. Otherwise, if $\sigma_1 < \tau$, then we construct a solution to (11.17) on the interval $[\sigma_1, \sigma_2)$ by solving (11.14) with initial condition $u(\sigma_1-) + \mathcal{K}(u(\sigma_1-), \Delta L(\sigma_1))$ and with noise restarted at time σ_1. Repeating this procedure inductively yields a local solution to (11.17).