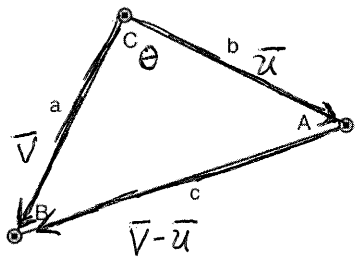
MATH 54 - WORKSHEET 9
GEOMETRY IN $\mathbb{R}^n$
FRIDAY 7/17

Work on these problems in groups of 3 or 4. Please discuss with your group and check each other's work. I'll be walking around the room checking in on various groups - if you have any questions, please ask! We will go over some of the answers together later in section.

(1) Recall that the Law of Cosines is a useful generalization of the Pythagorean Theorem. Given a triangle labeled as below, we have $a^2 + b^2 - 2ab\cos(C) = c^2$.



Thinking of the point at C as the origin, relabel the triangle above with vectors so that the segment from C to A is $\mathbf{u}$, the segment from C to B is $\mathbf{v}$, and the angle between these two vectors is θ. How should you label the segment from A to B?

(2) Using the relabeling from part (1), rephrase the Law of Cosines as an equation involving the dot product of vectors. Simplify and solve this equation for θ to get a formula for the angle between any two nonzero vectors in $\mathbb{R}^n$.

$$\|\bar{u}\|^2 + \|\bar{v}\|^2 - 2\|\bar{u}\|\|\bar{v}\|\cos(\theta) = \|\bar{v}-\bar{u}\|^2$$

$$\cancel{\bar{u}\cdot\bar{u}} + \cancel{\bar{v}\cdot\bar{v}} - 2\|\bar{u}\|\|\bar{v}\|\cos\theta = (\bar{v}-\bar{u})\cdot(\bar{v}-\bar{u}) = \cancel{\bar{v}\cdot\bar{v}} - 2\bar{u}\cdot\bar{v} + \cancel{\bar{u}\cdot\bar{u}}$$

$$-2\|\bar{u}\|\|\bar{v}\|\cos\theta = -2\bar{u}\cdot\bar{v}$$

$$\cos\theta = \frac{\bar{u}\cdot\bar{v}}{\|\bar{u}\|\|\bar{v}\|} \implies \theta = \cos^{-1}\left(\frac{\bar{u}\cdot\bar{v}}{\|\bar{u}\|\|\bar{v}\|}\right)$$

(3) Using the equation from part (2), find the angle between the following two vectors in $\mathbb{R}^4$:

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}.$$

$$\theta = \cos^{-1}\left(\frac{1\cdot 3 + 0\cdot 3 + 1\cdot 3 + 1\cdot 3}{\sqrt{1^2+0^2+1^2+1^2}\sqrt{3^2+3^2+3^2+3^2}}\right)$$

$$= \cos^{-1}\left(\frac{9}{\sqrt{3}\sqrt{36}}\right)$$

$$= \cos^{-1}\left(\frac{9}{\sqrt{3}\cdot 6}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6} \text{ radians}$$

$$(30^\circ)$$

(4) Find a single vector in $\mathbb{R}^4$ which is orthogonal to both vectors **u** and **v** from part (3).

If $\bar{u}\cdot\bar{x}=\bar{0}$ and $\bar{v}\cdot\bar{x}=\bar{0}$, then $\bar{x}$ is in $Nul(A)$ where $A=\begin{pmatrix}\bar{u}^T\\ \bar{v}^T\end{pmatrix}$, the matrix with rows $\bar{u}$ and $\bar{v}$.

Row reduce

$$\begin{pmatrix}1&0&1&1\\3&3&3&3\end{pmatrix}\longrightarrow\begin{pmatrix}1&0&1&1\\0&3&0&0\end{pmatrix}\rightarrow\begin{pmatrix}1&0&1&1\\0&1&0&0\end{pmatrix}$$

we find $x_1=-x_3-x_4$
$x_2=0$
x_3, x_4 free.

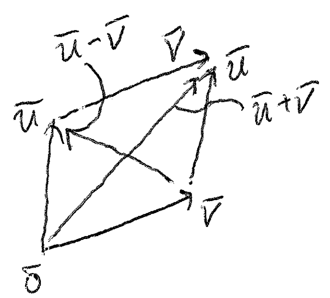
Anything in $\underbrace{Span\left\{\begin{pmatrix}-1\\0\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\0\\1\end{pmatrix}\right\}}_{\text{This is the orthogonal complement of } Span\{\bar{u},\bar{v}\}}$ works

(5) Prove the *parallelogram law*:

$$||\mathbf{u}+\mathbf{v}||^2+||\mathbf{u}-\mathbf{v}||^2=2||\mathbf{u}||^2+2||\mathbf{v}||^2$$

and draw a picture to explain why it is called the parallelogram law.

$$\underbrace{(\bar{u}+\bar{v})\cdot(\bar{u}+\bar{v})}_{||\bar{u}+\bar{v}||^2}+\underbrace{(\bar{u}-\bar{v})\cdot(\bar{u}-\bar{v})}_{||\bar{u}-\bar{v}||^2}=(\bar{u}\cdot\bar{u}+2\bar{u}\cdot\bar{v}+\bar{v}\cdot\bar{v})+(\bar{u}\cdot\bar{u}-2\bar{u}\cdot\bar{v}+\bar{v}\cdot\bar{v})$$
$$=2\bar{u}\cdot\bar{u}+2\bar{v}\cdot\bar{v}$$
$$=2||\bar{u}||^2+2||\bar{v}||^2$$



This says that the sum of the squares of the diagonal lengths of a parallelogram is equal to the sum of the squares of the side lengths.