MATH 54 - WORKSHEET 8
MATRIX POWERS AND DYNAMICAL SYSTEMS
WEDNESDAY 7/15

Work on these problems in groups of 3 or 4. Please discuss with your group and check each other's work. I'll be walking around the room checking in on various groups - if you have any questions, please ask! We will go over some of the answers together later in section.

In the problems below, consider the matrix $A = \begin{pmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{pmatrix}$.

It would be extremely tedious to calculate high powers of A, like $A^5 = AAAAA$ by matrix multiplication. In this worksheet, we'll develop a better method.

(1) I claim that $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are eigenvectors of A. Find their eigenvalues and use this information to write $P^{-1}AP = D$, where D is a diagonal matrix. Multiplying by P on the left and P^{-1} on the right, we also have $A = PDP^{-1}$.

$A\bar{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1\bar{v}_1$ So in the basis $B = \{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\}$, $[A_B] = \begin{pmatrix} 1 & 0 \\ 0 & 1/4 \end{pmatrix}$

$A\bar{v}_2 = \begin{pmatrix} 1/4 \\ -1/4 \end{pmatrix} = 1/4\, \bar{v}_2$ Then $D = P^{-1}AP$, where $D = \begin{pmatrix} 1 & 0 \\ 0 & 1/4 \end{pmatrix}$, $P = P_{E \leftarrow B} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$

and $P^{-1} = P_{B \leftarrow E} = \frac{1}{-3}\begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix}$, Also $PDP^{-1} = PP^{-1}APP^{-1} = A$.

(2) Compute D^2 and D^3 by matrix multiplication. Observing a pattern, write down a general formula for D^k when k is a natural number. [Moral: It's easy to take powers of diagonal matrices.]

$$D^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1/4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1/4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/16 \end{pmatrix}$$

$$D^3 = D^2 D = \begin{pmatrix} 1 & 0 \\ 0 & 1/16 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1/4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/64 \end{pmatrix}$$

In general, $D^k = \begin{pmatrix} 1 & 0 \\ 0 & (1/4)^k \end{pmatrix}$

In fact, if $A = (\bar{a}_1 \dots \bar{a}_n)$ is any matrix, and $D = (\bar{d}_1 \dots \bar{d}_n)$ is diagonal, so $\bar{d}_i = \lambda_i \bar{e}_i$, then $AD = (A\bar{d}_1 \dots A\bar{d}_n) = (\lambda_1 A\bar{e}_1 \dots \lambda_n A\bar{e}_n) = (\lambda_1 \bar{a}_1 \dots \lambda_n \bar{a}_n)$. So the columns of A are multiplied by the diagonal entries of D. Then it's easy to see that if $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$, $D^k = \begin{pmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{pmatrix}$

(3) Show that $A^k = PD^kP^{-1}$ for any natural number k. Use this information to write down a general formula for A^k.

$$A^k = \underbrace{(PDP^{-1})(PDP^{-1})\dots(PDP^{-1})}_{K \text{ times}} = P\underbrace{DD\dots D}_{K \text{ times}}P^{-1} = PD^kP^{-1}$$

adjacent P and P^{-1} cancel.

$$\text{So } A^k = \underset{P}{\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}} \underset{D^k}{\begin{pmatrix} 1 & 0 \\ 0 & (1/4)^k \end{pmatrix}} \underset{P^{-1}}{\begin{pmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{pmatrix}} = \begin{pmatrix} 1 & (1/4)^k \\ 2 & -(1/4)^k \end{pmatrix}\begin{pmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{pmatrix} = \begin{pmatrix} 1/3 + 2/3(1/4)^k & 1/3 - 1/3(1/4)^k \\ 2/3 - 2/3(1/4)^k & 2/3 + 1/3(1/4)^k \end{pmatrix}$$

Given a function $T\colon V \to V$, we can study the *dynamical system* defined by T: the behavior of vectors in V under repeated applications of T.

$$\mathbf{v} \mapsto T(\mathbf{v}) \mapsto T(T(\mathbf{v})) \mapsto \dots$$

(4) **The grass is always greener:** A society of immortal mathematicians resides in the magical realm of Linearia. Linearia is divided into two regions, Algebraistan and the Republic of Differential Equations (RDE). The mathematicians, being immortal, get rather bored, and so they move around frequently. Each year, half of the residents of RDE decide to move to Algebraistan. But since Algebraistan is rather a more interesting place, only one quarter of the residents of Algebraistan decide to move to RDE each year.

If the population Linearia in one year is represented as a vector $\mathbf{v} = \begin{pmatrix} r \\ a \end{pmatrix}$, where r is the population of RDE and a is the population of Algebraistan, explain why the popluation the next year is given by $A\mathbf{v}$, where $A = \begin{pmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{pmatrix}$ is the matrix on the other side of the worksheet.

If r = pop. of RDE and a = pop. of Alg., then next year the population of RDE is $\underbrace{\tfrac{1}{2}r}_{\text{those that stuck around}} + \underbrace{\tfrac{1}{4}a}_{\text{new comers}}$ and the population of Alg. is $\underbrace{\tfrac{3}{4}a}_{\text{those that stuck around}} + \underbrace{\tfrac{1}{2}r}_{\text{newcomers}}$.

This is represented by the matrix multiplication $\underbrace{\begin{pmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} r \\ a \end{pmatrix}}_{\vec{v}} = \underbrace{\begin{pmatrix} \tfrac{1}{2}r + \tfrac{1}{4}a \\ \tfrac{1}{2}r + \tfrac{3}{4}a \end{pmatrix}}_{A\vec{v}}$

(5) Given a starting population $\mathbf{v} = \begin{pmatrix} r \\ a \end{pmatrix}$, we may wish to know how many residents live in each region after k years. Explain why this is described by $A^k\mathbf{v}$, and write down a general formula for this vector, using the formula for A^k from part (3).

If multiplication by A gives the population after one year has passed, the population after k years is $\underbrace{A(A(\dots A(\vec{v}))\dots))}_{k \text{ times}} = A^k\vec{v}$.

$$A^k\vec{v} = \begin{pmatrix} \tfrac{1}{3} + \tfrac{2}{3}(\tfrac{1}{4})^k & \tfrac{1}{3} - \tfrac{1}{3}(\tfrac{1}{4})^k \\ \tfrac{2}{3} - \tfrac{2}{3}(\tfrac{1}{4})^k & \tfrac{2}{3} + \tfrac{1}{3}(\tfrac{1}{4})^k \end{pmatrix}\begin{pmatrix} r \\ a \end{pmatrix} = \begin{pmatrix} \tfrac{1}{3}(r+a) + (\tfrac{1}{4})^k(\tfrac{2}{3}r - \tfrac{1}{3}a) \\ \tfrac{2}{3}(r+a) + (\tfrac{1}{4})^k(\tfrac{1}{3}a - \tfrac{2}{3}r) \end{pmatrix}$$

(6) Examine the limiting behavior of this system by computing $\lim_{k\to\infty} A^k\mathbf{v}$. What does this tell you about the populations of Algebraistan and the RDE after many years have passed? How does the answer depend on the starting populations r and a?

$\lim_{k\to\infty} A^k\vec{v} = \begin{pmatrix} \tfrac{1}{3}(r+a) \\ \tfrac{2}{3}(r+a) \end{pmatrix}$ since $\lim_{k\to\infty}(\tfrac{1}{4})^k = 0$.

After many years have passed, about $\tfrac{2}{3}$ of the total population of Linearia will live in Algebraistan, and $\tfrac{1}{3}$ will live in RDE.

The original population distribution $\vec{v} = \begin{pmatrix} r \\ a \end{pmatrix}$ didn't matter at all!

All that mattered was the total population $(r+a)$.