MATH 54 - WORKSHEET 7
COORDINATES AND MATRICES
TUESDAY 7/7

Work on these problems in groups of 3 or 4. Please discuss with your group and check each other's work. I'll be walking around the room checking in on various groups - if you have any questions, please ask! We will go over some of the answers together later in section.

Given a basis $B = \{\mathbf{b_1}, \dots, \mathbf{b_n}\}$ for a vector space V, we defined the B-coordinate map $[\cdot]_B \colon V \to \mathbb{R}^n$, which allows us to represent vectors $\mathbf{v}$ in V as vectors $[\mathbf{v}]_B$ in $\mathbb{R}^n$. In this worksheet, we will see how to use coordinates to respresent changes of basis and linear transformations as matrices.

Change of Basis [Section 4.7 of the textbook]

Let $B = \{\mathbf{b_1}, \dots, \mathbf{b_n}\}$ and $C = \{\mathbf{c_1}, \dots, \mathbf{c_n}\}$ be two bases for the same vector space V. We define the *change of basis matrix from B to C*:

$$\underset{C\leftarrow B}{P} = \begin{pmatrix} [\mathbf{b_1}]_C & \cdots & [\mathbf{b_n}]_C \end{pmatrix}$$

That is, $\underset{C\leftarrow B}{P}$ is an $n \times n$ matrix whose columns are obtained by taking each vector $\mathbf{b_i}$ in B writing it in C-coordinates.

(1) Let B be the standard basis for $\mathbb{R}^2$, $B = \{\mathbf{e_1}, \mathbf{e_2}\}$. Consider a new basis for $\mathbb{R}^2$, $C = \{\mathbf{c_1}, \mathbf{c_2}\}$, where $\mathbf{c_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{c_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Using the definition above, write down the matrix $\underset{C\leftarrow B}{P}$.

$[\bar{e}_1]_C = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ since $\vec{e}_1 = 0\bar{c}_1 + 1\bar{c}_2$

$[\bar{e}_2]_C = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$ since $\bar{e}_2 = 1/2\,\bar{c}_1 - 1/2\,\bar{c}_2$

$\underset{C\leftarrow B}{P} = ([\bar{e}_1]_C \; [\bar{e}_2]_C) = \begin{pmatrix} 0 & 1/2 \\ 1 & -1/2 \end{pmatrix}$

(2) Let $\mathbf{v} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$. Write down $[\mathbf{v}]_B$ (this is not hard) and $[\mathbf{v}]_C$ (for this one you need to think a little more). Using the matrix $\underset{C\leftarrow B}{P}$ you found above, check that $[\mathbf{v}]_C = \underset{C\leftarrow B}{P}[\mathbf{v}]_B$.

$[\bar{v}]_B = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ since $\bar{v} = 4\bar{e}_1 + 4\bar{e}_2$

$[\bar{v}]_C = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ since $\bar{v} = 2\bar{c}_1 + 2\bar{c}_2$

And $\underbrace{\begin{pmatrix} 2 \\ 2 \end{pmatrix}}_{[\bar{v}_C]} = \underbrace{\begin{pmatrix} 0 & 1/2 \\ 1 & -1/2 \end{pmatrix}}_{\underset{C\leftarrow B}{P}} \underbrace{\begin{pmatrix} 4 \\ 4 \end{pmatrix}}_{[\bar{v}_B]}$ ✓

(3) Show in general that for any $\mathbf{v}$ in a vector space V, we have $[\mathbf{v}]_C = \underset{C\leftarrow B}{P}[\mathbf{v}]_B$.

If $[\bar{v}]_B = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, then $\underset{C\leftarrow B}{P}[\bar{v}_B] = ([\bar{b}_1]_C \dots [\bar{b}_n]_C)\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

so $\bar{v} = a_1\bar{b}_1 + \dots + a_n\bar{b}_n$

$= a_1[\bar{b}_1]_C + \dots + a_n[\bar{b}_n]_C$

$= [a_1\bar{b}_1 + \dots + a_n\bar{b}_n]_C$ since the coordinate map $[\cdot]_C$ is linear.

$= [\bar{v}]_C$

Linear Transformations [Section 5.4 of the textbook]

Let $B = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ be a basis for the vector space V, and let $C = \{\mathbf{c_1}, \ldots, \mathbf{c_m}\}$ be a basis for the vector space W. Given a linear transformation $T: V \to W$, we define the *matrix for T relative to the bases B and C*:

$$[T]_C^B = \big(\ [T(\mathbf{b_1})]_C \ \ \cdots \ \ [T(\mathbf{b_n})]_C \ \big)$$

That is, $[T]_C^B$ is an $m \times n$ matrix whose columns are the images under T of the basis vectors in B, written in C-coordinates.

(1) Consider the linear transformation $\frac{d}{dx}: \mathbb{P}_{\leq 3} \to \mathbb{P}_{\leq 2}$. Take $B = \{1, x, x^2, x^3\}$ as a basis for $\mathbb{P}_{\leq 3}$ and $C = \{1, x, x^2\}$ as a basis for $\mathbb{P}_{\leq 2}$. Using the definition above, write the matrix $[\frac{d}{dx}]_C^B$ for $\frac{d}{dx}$ relative to the bases B and C.

$$\left[\tfrac{d}{dx}\right]_C^B = \left(\left[\tfrac{d}{dx}1\right]_C \ \left[\tfrac{d}{dx}x\right]_C \ \left[\tfrac{d}{dx}x^2\right]_C \ \left[\tfrac{d}{dx}x^3\right]_C \right)$$

$$= \left([0]_C \ [1]_C \ [2x]_C \ [3x^2]_C \right)$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

(2) Let $p(x)$ be the polynomial $1 + 2x + -x^2 + x^3$. Write down $[p(x)]_B$ and $[\frac{d}{dx}p(x)]_C$. Using the matrix for $[\frac{d}{dx}]_C^B$ you found above, check that $[\frac{d}{dx}p(x)]_C = [\frac{d}{dx}]_C^B [p(x)]_B$.

$$[p(x)]_B = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix} \qquad \left[\tfrac{d}{dx}p(x)\right]_C = [2 - 2x + 3x^2]_C = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$$

And

$$\underbrace{\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}}_{[\frac{d}{dx}p(x)]_C} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}}_{[\frac{d}{dx}]_C^B} \underbrace{\begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix}}_{[p(x)]_B}$$

(3) Show in general that for any $\mathbf{v}$ in V, we have $[T(\mathbf{v})]_C = [T]_C^B[\mathbf{v}]_B$.

If $[\bar{v}]_B = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, then $[T]_C^B[\bar{v}]_B = \left([T(\bar{b}_1)]_C \ldots [T(\bar{b}_n)]_C \right) \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

so $\bar{v} = a_1\bar{b}_1 + \ldots + a_n\bar{b}_n$

$= a_1[T(\bar{b}_1)]_C + \ldots + a_n[T(\bar{b}_n)]_C$

$= [a_1T(\bar{b}_1) + \ldots + a_nT(\bar{b}_n)]_C$ — since the coordinate map $[\cdot]_C$ is linear

$= [T(a_1\bar{b}_1 + \ldots + a_n\bar{b}_n)]_C$ — since T is linear

$= [T(\bar{v})]_C$